Quiz III

STAT 305, Section 3
FALL 2019

Instructions

- The quiz is scheduled for 80 minutes, from 09:30 to 10:50 AM. At 10:50 AM the exam will end.
- Total points for the exam is 60. Points for individual questions are given at the beginning of each problem. Show all your calculations clearly to get full credit. Put final answers in the box at the right (except for the diagrams!).
- A forumula sheet is attached to the end of the exam. Feel free to tear it off.
- Normal quantile table is attached to the end of the exam. Feel free to tear it off.
- You may use a calculator during this exam.
- Answer the questions in the space provided. If you run out of room, continue on the back of the page.
- If you have any questions about, or need clarification on the meaning of an item on this exam, please ask your instructor. No other form of external help is permitted attempting to receive help or provide help to others will be considered cheating.
- **Do not cheat on this exam.** Academic integrity demands an honest and fair testing environment. Cheating will not be tolerated and will result in an immediate score of 0 on the exam and an incident report will be submitted to the dean's office.

Name: Key

Student ID: ______________________

1. Suppose a standup comedian plans to give a total of $n = 5$ jokes in an entire 2-hour performance. Call a joke a succes if at least one audience member laughs. If no audience member laughs, the joke is a failure. Assume that all the jokes are equally funny, with $p = p(success) = 0.2$. Let X be the random variable that denotes the number of jokes out of the total 5 were successes.

   (a) (3 points) Precisely state the distribution of X, giving the values of any parameters necessary.

   $X \sim \text{binomial}(n=5, p=0.2)$

   (b) (3 points) Calculate the probability that the whole night is a failure: i.e., $P(no\ laughs)$

   $$P(x=0) = \binom{5}{0}(0.2)^0(1-0.2)^{5-0}$$
   $$= (0.8)^5$$

   (c) (3 points) Caculate the probability that the comedian tells at least 4 successful jokes.

   $$P(X \geq 4) = P(X=4) + P(X=5)$$
   $$= \binom{5}{4}(0.2)^4(0.8)^1 + \binom{5}{5}(0.2)^5(0.8)^0$$
   $$= 0.0064 + 0.00032 = 0.00672$$

   (d) (3 points) Calculate the expected number of successfull jokes.

   In Binomial distribution:

   $E(X) = n.p = 5(0.2) = 1$

   (This means it is expected that one of his jokes is successful

   (e) (3 points) Calculate the standard deviation of X

in Binomial: $Var(X) = nP(1-P) = 5(0.2)(1-0.2) = 0.8$

$$SD(X) = \sqrt{Var(X)} = \sqrt{0.8} = 0.89$$

1. Let X be a normal random variable with a mean of 2 and a varaince of 9 (i.e., $X \sim N(2, 9)$) and let Z be a random variable following a standard normal distribution.

   (a) Find the following probabilities (note: Table B-3 may be helpful):

   i. (2 points) $P(Z \leq 2)$

   $$P(Z \leq 2) = \Phi(2) = 0.9772$$

   ii. (2 points) $P(|Z| \geq 1)$

   $$P(|Z| > 1) = P(Z > 1) + P(Z < -1) = 2\Phi(-1) = 0.3173$$

   iii. (2 points) $P(0 \leq Z < 3)$

   $$= \Phi(3) - \Phi(0) = 0.9986 - 0.5 = 0.4986$$

   iv. (2 points) $P(X < 3)$

   $$= P\left(\frac{X-2}{3} < \frac{3-2}{3}\right) = P(Z < 1/3) = \Phi(0.333) = 0.6305$$

   v. (2 points) $P(|X| \leq 4.5)$

   $$= P(-4.5 < X < 4.5) = P\left(\frac{-4.5-2}{3} < Z < \frac{4.5-2}{3}\right) = P\left(-\frac{6.5}{3} < Z < \frac{2.5}{3}\right)$$

   $$= (-2.16 < Z < 0.833) = \Phi(0.833) - \Phi(-2.16) = 0.7821$$

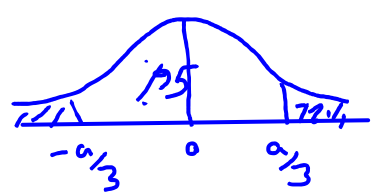
   (b) (5 points) Find the value a so that $P(-a + 2 < X < a + 2) = .95$ (approximate as needed).

$$0.95 = P(-a+2 < X < a+2) = P\left(\frac{-a+2-2}{3} < Z < \frac{a+2-2}{3}\right)$$

$$= P\left(-\frac{a}{3} < Z < \frac{a}{3}\right)$$

$$= P(a/3) - P(-a/3)$$

$$= \Phi(a/3) - \Phi(-a/3)$$



$$= 1 - \Phi(-a/3) - \Phi(-a/3)$$

$$= 1 - 2\Phi(-a/3)$$

$$\Rightarrow \Phi(-a/3) = \frac{0.05}{2} = 0.025 \Rightarrow -\frac{a}{3} = -1.96 \Rightarrow \boxed{a = 5.88}$$

2. Suppose that X is a continuous random variable with probability density function (pdf):

$$f(x) = \begin{cases} 0 & x < 0 \\ cx^2 & -2 < x < 2 \\ 0 & x \geq 2 \end{cases}$$

where c is a constant.

(a) (2 points) What is the value of c if $f(x)$ is a valid probability density function?

$1 = \int_{-2}^{2} cx^2 \, dx = 2c \int_0^2 x^2 \, dx = 2c \cdot \frac{x^3}{3} \Big|_0^2 = 2c \left(\frac{8}{3}\right)$

$\Rightarrow c = 3/16$

(b) (4 points) What is the cumulative density function, $F(x)$?

$F(x) = P(X \leq t) = \int_{-2}^{t} \frac{3}{16} x^2 \, dx = \frac{3}{16} \cdot \frac{x^3}{3} \Big|_{-2}^{t}$

$= \frac{1}{16} x^3 \Big|_{-2}^{t} = \frac{1}{16} (t^3 + 8) = \frac{t^3}{16} + \frac{1}{2}$

(c) (2 points) What is the probability that X takes a value greater than 1?

$P(X > 1) = \int_1^2 \frac{3}{16} x^2 \, dx = \cdots$

or $= 1 - P(X \leq 1) = 1 - F(1) = 1 - \left[\frac{1^3}{16} + \frac{1}{2}\right] = \frac{7}{16}$

(d) (2 points) What is the probability that X takes a value between 0 and 1?

$P(0 < X < 1) = \int_0^1 \frac{3}{16} x^2 \, dx = \frac{1}{16} x^3 \Big|_0^1 = 1/16$

3. Let X, Y and Z be random variables with expected value and standard deviations given below:

	Expected Value	Standard Deviation
X	1.5	3.2
Y	0	8.1
Z	6	2.7

(a) (3 points) Find $E(8+2X+Y-Z)$ = $8+2EX+EY-EZ$

$= 8+2(1.5)+0-6$

$= 8+3-6 = 5$

(b) (3 points) Find $Var(\frac{Z}{3}-2Y+3X)$ = $\frac{1}{9}var(Z)+4var(Y)+9\ var(X)$

$= \frac{1}{9}(2.7^2)+4(8.1^2)+9(3.2^2)$

$= 355.41$

4. Let X be the the number of crankshafts that fail in a given test of a certain type of vehicle ($X = 0, 1, 2$). Let $Y = 1$ if the clutch fails during that same test and $Y = 0$ otherwise. Consider the joint distribution of X and Y :

Y \X	0	1	2
0	0.35	0.1	0.05
1	0.2	0.25	0.05

(a) (2 points) Find $P(X = 1, Y = 1)$ = $f(1,1) = 0.25$

(b) (2 points) $P(x > 0$ and $Y = 1)$ = $P(X=1, Y=1) + P(X=2, Y=1)$

$= f(1,1) + f(2,1) = 0.25 + 0.05 = 0.3$

(c) (2 points) Find the marginal pmfs of X and Y

x	0	1	2
f(x)	0.55	0.35	0.1

y	0	1
f(y)	0.5	0.5

(d) (2 points) Find the conditional distribution of $f_{X|Y}(x|y=1)$

x	0	1	2
$f_{X\|Y}(x\|y=1)$	0.4	0.5	0.1

$f_{X|Y}(0|1) = \frac{f(0,1)}{f_Y(1)} = \frac{0.2}{0.5} = 0.4$

$f_{X|Y}(1|1) = \frac{f(1,1)}{f_Y(1)} = \frac{0.25}{0.5} = 0.5$

$f_{X|Y}(2|1) = \frac{f(2,1)}{f_Y(1)} = \frac{0.05}{0.5} = 0.1$

(e) (2 points) Are X and Y independent? Why or why not?

No, because $0.35 = f_{X,Y}(0,0) \neq f_X(0)\, f_Y(0) = 0.55 \times 0.5$
$= 0.275$

5. (4 points) Let $X_1, X_2, \cdots, X_{41}$ are iid random variables with Exponential distribution with parameter 1/4. (i.e $\sim$ Exp($\alpha = 1/4$)). Find the probability that $P(|\overline{X} - 1| < 2)$.

$n = 41 \geq 25 \Rightarrow$ we can use CLT:

$EX_i = \alpha = 1/4 = 0.25$

$Var(X_i) = \alpha^2 = 1/16$, $SD(X) = 1/4 = 0.25$

$n = 41$, $\bar{X} \sim N(0.25, \sigma = \frac{0.25}{\sqrt{41}} = 0.039)$ → $SD(\bar{X})$

Recall: $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \dot\sim N(0,1)$

$P(|\bar{x} - 1| < 2) = P(-2 < \bar{x} - 1 < 2)$

$= P(-1 < \bar{x} < 3)$

$= P\left(\frac{-1-0.25}{0.039} < Z < \frac{3-0.25}{0.039}\right)$

$SD(\bar{X}) = \frac{\sigma}{\sqrt{n}}$

$$= P\left(\frac{-1.25}{0.039} < Z < \frac{2.75}{0.039}\right)$$

$$= P(-32.051 < Z < 70.512)$$

$$= \Phi(70.512) - \Phi(-32.051)$$

$$= 1 - 0 = 1$$