STAT 305: Chapter 5

Part II

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Discrete Random Variables

Meaning, Use, and Common Distributions

General Info About Discrete RVs

Reminder: What is a Random Variable?

Random Variables, we have already defined, take real-numbered ($\mathbb{R}$) values based on outcomes of a random experiment.

- If we know the outcome, we know the value of the random variable (so that isn't the random part).
- However, before we perform the experiment we do not know the outcome - we can only make statements about what the outcome is likely to be (i.e., we make "probabilistic" statements).
- In the same way, we do not know the value of the random variable before the experiment, but we can make probability statements about what value the RV might take.

General Info About Discrete RVs

Reminder: What is a Random Variable?

General Info

Reminder: RVs

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Terms & Notation

Common Terms and Notation for Discrete RVs

Of course, we can't introduce a *sort of* new concept without introducing a whole lot of new terminology.

We use capital letters to refer to discrete random variables: X, Y, Z, ...

We use lower case letters to refer to values the discrete RVs can take: x, x_1, y, z, ...

While we can use $P(X = x)$ to refer to the probability that the discrete random variable takes the value x, we usually use what we call the **probability function**:

- For a discrete random variable X, the probability function $f(x)$ takes the value $P(X = x)$
- In otherwords, we just write $f(x)$ instead of $P(X = x)$.

random variable

rolling a die : X = a r.v associated with the number that faces up.

values (

$x = \{1, 2, 3, 4, 5, 6\}$

General Info

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Using **random variables** we can translate **events** into the world of mathematics and probability. So, we can then express event occuring in the language of **probability**

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Common Terms and Notation for Discrete RVs

We also have another function related to the probabilities, called the **cumulative probability function**.

For a discrete random variable X taking values $x_1, x_2, \dots$ the CDF or **cumulative probability function** of X, $F(x)$, is defined as

$$\text{CDF} \quad F(x) = \sum_{z \leq x} f(z) \quad \text{PMF}$$

Which in other words means that for any value x,

$$\text{PMF} \longrightarrow f(x) = P(X = x)$$

and

$$\text{CDF} \equiv F(x) = P(X \leq x) = \sum_{Z \leq x} f(Z)$$

e.g. : rolling a die:

$$F(4) = P(X \leq 4) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$$

$= 4/6$

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Note that

> Discrete: Probability Mass Function (pmf)
>
> Continuous: Probability Density Function (pdf)

A probability mass function (pmf) gives probabilities of occurance for individual values.

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> i.e. a dicrete random variable X takes individual (discrete) values in an interval.

e.g. Rolling a die:

$$P(X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3)$$

or

$$P(2 < X \leq 4) = P(3 \leq X \leq 4) = P(X = 3) + P(X = 4)$$

but

$$P(X \in (2, 3)) = P(2 < X < 3) = 0$$

or

$$P(3 < X \leq 4) = P(X = 4)$$

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Properties of a valid probability function.

In order to have a valid pmf we need

$0 \leq P(X=x) \leq 1$

> 1) $f(x) \geq 0$ for $\forall x \in S$
>
> 2) $\sum_x f(x) = 1$ (probabilties sum to one)

Example

Let $f(x)$ be a pmf defined as

$$f(x) = p(X = x) = \begin{cases} 1/8 & x = -1, 1 \\ a & x = 0 \\ 1/4 & x = -2, 2 \end{cases}$$

What value of **a** makes $f(x)$ a valid pmf?

① $a \geq 0$

② $\sum_{x \in \{-2,-1,0,1,2\}} f(x) = 1$

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Properties of a valid probability function.

Example

$$f(x) = p(X = x) = \begin{cases} 1/8 & x = -1, 1 \\ a & x = 0 \\ 1/4 & x = -2, 2 \end{cases} \rightarrow 1/4$$

1) $f(x) \geq 0 \forall x \in (-2, -1, 0, 1, 2) \Rightarrow a \geq 0$

2) $\sum_{x \in (-2,-1,0,1,2)} f(x) = 1$. so,

$$1 = f(-1) + f(1) + f(0) + f(2) + f(-2)$$

$$= 1/8 + 1/8 + a + 1/4 + 1/4$$

$$= a + 2/8 + 1/2$$

which gives us $a = 1/4$

Common Terms and Notation for Discrete RVs (cont)

The values that X can take and the probabilities attached to those values are called the **probability distribution** of X (since we are talking about how the total probability 1 gets spread out on (or distributed to) the values that X can take).

Example

Suppose that the we roll a die and let T be the number of dots facing up.

$p(t)$

Define the probability distribution of T. Find $f(3)$, $P(3 < \cancel{X}\, T \leq 6)$, $F(3)$ and $F(6)$.

$= P(T = 3)$

$P(T \leq 6)$

$P(T \leq 3)$

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Recall: **pmf**: $f(x) = P(X = x)$

CDF: $F(x) = P(X \leq x)$

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We can write write the pmf and CDF of T in a table like

T	1	2	3	4	5	6
f(T)	1/6	1/6	1/6	1/6	1/6	1/6
F(T)	1/6	2/6	3/6	4/6	5/6	1

$P(T \leq t) =$

$P(T \leq 1) = P(T=1)$

$F(5) = P(T \leq 5)$

- $P(T=3) \equiv$ $f(3) = 1/6$
- $P(3 < T \leq 6) = \boxed{P(4 \leq T \leq 6)}$
  $= f(4) + f(5) + f(6)$
  $= 1/6 + 1/6 + 1/6 = 3/6$
- $F(3) = P(T \leq 3) = f(1) + f(2) + f(3)$
  $= 1/6 + 1/6 + 1/6 = 3/6$
- $F(6) = P(T \leq 6) = \sum_{t=1}^{6} f(T) = 1$

General Info

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Discrete? $P(Z=z)=$

Terms & Notation

Example: [Torque]

Let $Z =$ the torque, rounded to the nearest integer, required to loosen the next bolt on an apparatus.

Z	11	12	13	14	15	16	17	18	19	20
f(z)	0.03	0.03	0.03	0.06	0.26	0.09	0.12	0.20	0.15	0.03

Calculate the following probabilities:

$F(14) =$
- $P(Z \leq 14)$
- $P(Z > 12)$
- $P(Z \text{ is even})$
- $P(Z \in \{15, 16, 18\})$

Example: [Torque]

Z	11	12	13	14	15	16	17	18	19	20
f(z)	0.03	0.03	0.03	0.06	0.26	0.09	0.12	0.20	0.15	0.03

$$
\begin{aligned}
P(Z \leq 14) &= P(Z = 11 \text{ or } Z = 12 \text{ or } Z = 13 \text{ or } Z = 14) \\
&= P(Z = 11) + P(Z = 12) + P(Z = 13) + P(Z = 14) \\
&= 0.03 + 0.03 + 0.03 + 0.06 \\
&= 0.15
\end{aligned}
$$

$$
\begin{aligned}
P(Z > 12) = P(&Z = 13 \text{ or } Z = 14 \text{ or } Z = 15 \text{ or } Z = 16 \\
&\text{or } Z = 17 \text{ or } Z = 18 \text{ or } Z = 19 \text{ or } Z = 20) \\
&= P(Z = 13) + P(Z = 14) + P(Z = 15) + P(Z = 16) \\
&+ P(Z = 17) + P(Z = 18) + P(Z = 19) + P(Z = 20) \\
&= 0.03 + 0.03 + 0.06 + 0.26 + 0.09 + 0.12 + 0.2 + 0.15 + 0.03 \\
? \quad &= 0.9\cancel{7}4
\end{aligned}
$$

Example: [Torque]

Z	11	12	13	14	15	16	17	18	19	20
f(z)	0.03	0.03	0.03	0.06	0.26	0.09	0.12	0.20	0.15	0.03

That is a very long way. A smarter way is to use **complement** probability.

$$
\begin{aligned}
P(Z > 12) &= 1 - P(Z \le 12) \\
&= 1 - \{P(Z = 11) + P(Z = 12)\} \\
&= 1 - \{0.03 + 0.03\} \\
&= \cancel{0.97}
\end{aligned}
$$

$1 - 0.06 = 0.94$

$$
\begin{aligned}
P(Z \text{ is even}) &= P(Z = 12 \text{ or } Z = 14 \text{ or } Z = 16 \text{ or } Z = 18 \text{ or } Z = 20) \\
&= P(Z = 12) + P(Z = 14) + P(Z = 16) \\
&\quad + P(Z = 18) + P(Z = 20) \\
&= 0.03 + 0.06 + 0.09 + 0.2 + 0.03 \\
&= \boxed{0.41}
\end{aligned}
$$

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Example: [Torque]

Z	11	12	13	14	15	16	17	18	19	20
f(z)	0.03	0.03	0.03	0.06	0.26	0.09	0.12	0.20	0.15	0.03

$$
\begin{aligned}
P(Z \in \{15, 16, 18\}) &= P(Z = 15 \text{ or } Z = 16 \text{ or } Z = 18) \\
&= P(Z = 15) + P(Z = 16) + P(Z = 18) \\
&= 0.26 + 0.09 + 0.2 \\
&= 0.55
\end{aligned}
$$

More on Cumulative Probability Function (CDF)

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More on CDF

> The *cumulative probability distribution (cdf)* for a random variable X is a function $F(x)$ that for each number x gives the probability that X takes that value or a smaller one, $F(x) = P[X \leq x]$.

Since (for discrete distributions) probabilities are calculated by summing values of $f(x)$,

$$F(x) = P[X \leq x] = \sum_{y \leq x} f(y)$$

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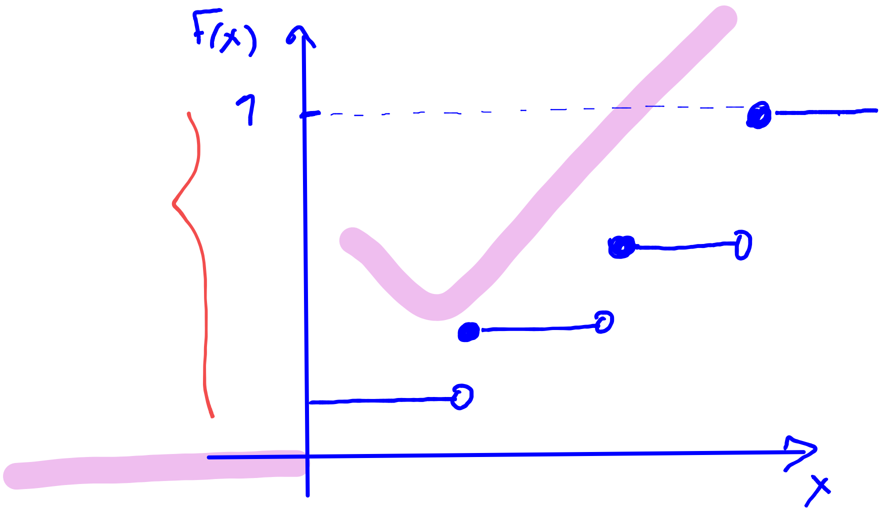
More on CDF

Properties of a mathematically valid cumulative distribution function:

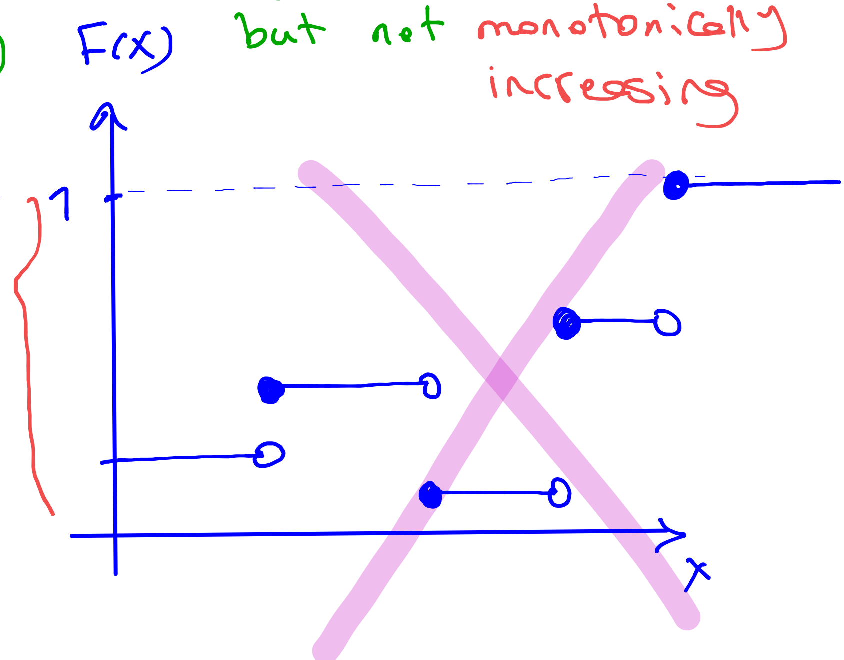
- $F(x) \geq 0$ for all real numbers x
- $F(x)$ is monotonically **increasing**
- $F(x)$ is right continuous
- $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$ $\equiv$ $0 \leq F(x) \leq 1$
  - This means that $0 \leq F(x) \leq 1$ for **any CDF**

> In the discrete cases, the graph of $F(x)$ will be n stair-step graph with jumps at possible values of our random variable and height equal to the probabilities associated with those values

Right continuous & monotonically increasing



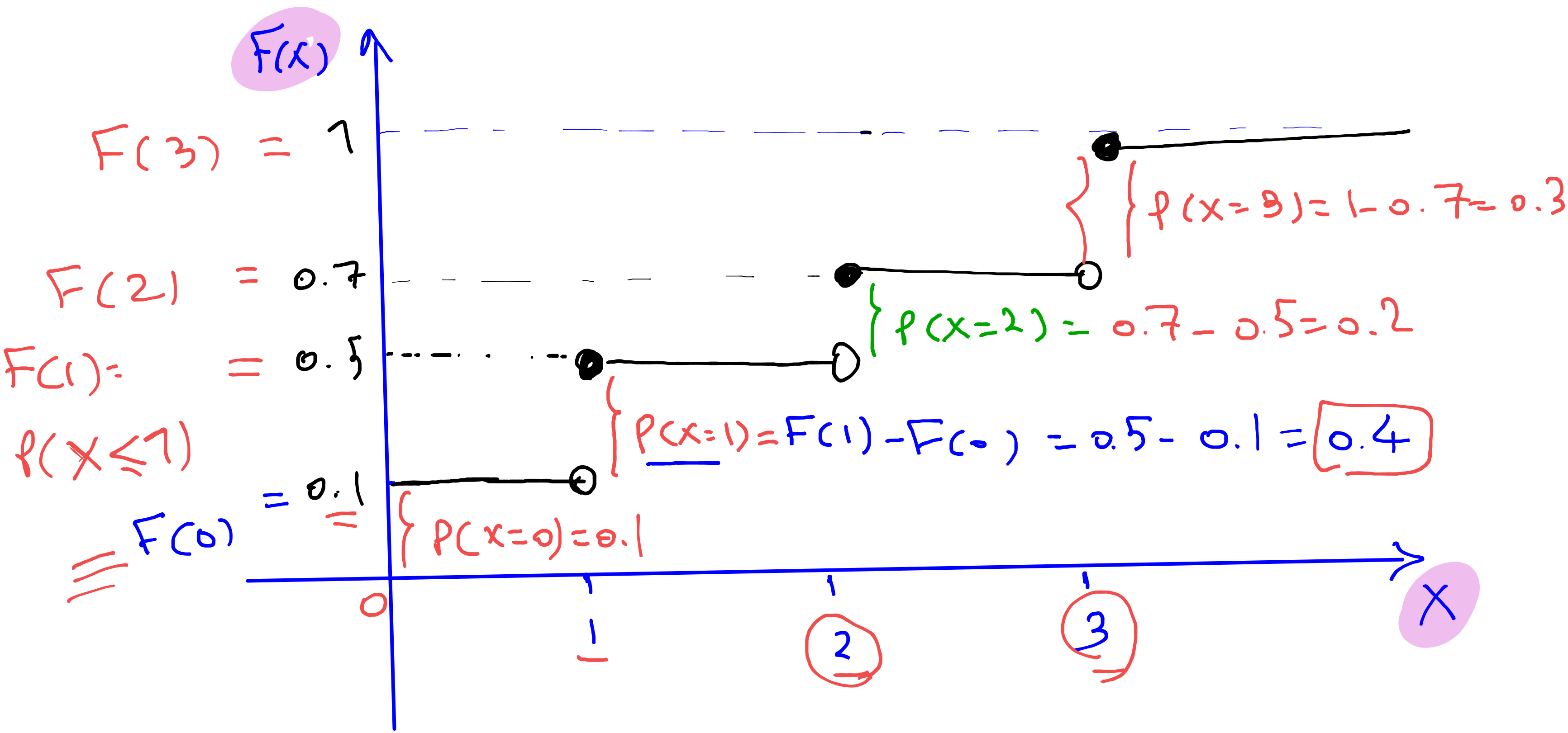
Right continuous, but not monotonically increasing



always: $0 \leq F(x) \leq 1$

Note: The height of each step (jump) is the probability $P(X = x)$ in

Discrete CDF:



$$f(x) = \begin{cases} 0.1 & x = 0 \\ 0.4 & x = 1 \\ 0.2 & x = 2 \\ 0.3 & x = 3 \end{cases} \quad = F(x) = \begin{cases} 0.1 & x = 0 \\ 0.5 & x = 1 \\ 0.7 & x = 2 \\ 1 & x = 3 \end{cases}$$

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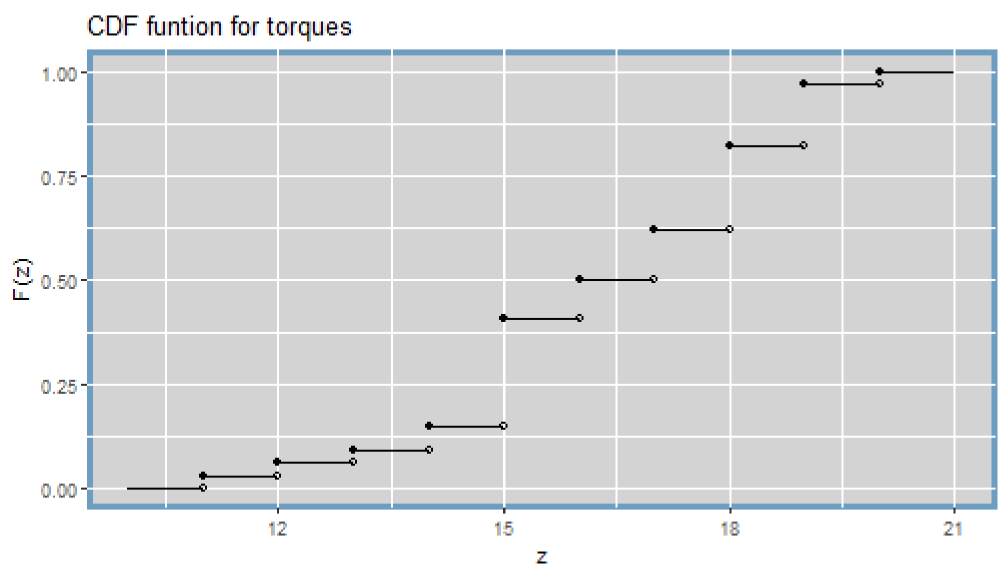
More on CDF

Example: [Torque] Let $Z =$ the torque, rounded to the nearest integer, required to loosen the next bolt on an apparatus.

Z	11	12	13	14	15	16	17	18	19	20
f(z)	0.03	0.03	0.03	0.06	0.26	0.09	0.12	0.20	0.15	0.03



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More on CDF

Calculate the following probabilities using the **cdf only**:

- $F(10.7)$
- $P(Z \leq 15.5)$
- $P(12.1 < Z \leq 14)$
- $P(15 \leq Z < 18)$

More on CDF

One more example

Say we have a random variable Q with pmf:

q	f(q)
1	0.34
2	0.10
3	0.22
7	0.34

Draw the CDF

Summaries

Almost all of the devices for describing relative frequency (empirical) distributions in Ch. 3 have versions that can describe (theoretical) probability distributions.

1. Measures of location == Mean

2. Measures of spread == variance

3. Histogram == probability histograms based on theoretical probabilities

Mean

and

Variance

of Discrete Random Variables

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Mean of a Discrete Random Variable

For a discrete random varable, X, which can take values $x_1, x_2, \ldots$ we define **the mean of X** (also known as **the expected value of X**) as:

$$E(X) = \sum_{i=1}^{n} x_i \cdot f(x_i)$$

We often use the symbol μ instead of $E(X)$.

Also, just to be confusing, you will often see EX instead of $E(X)$. Use context clues.

Example:

Suppose that the we roll a die and let T be the number of dots facing up. Find the expected value of T.

An intuitive way to get Expected value and variance of any random variable:

For discrete r.v: say we have a pmf $f(x)$:

$$E(\text{"anything"}) = \sum_{\forall x \in S} (\text{value of "anything"}) \cdot f(x)$$

e.g:

$$E(X) = \sum_{x \in S} x f(x) = \sum_{x \in S} x \cdot P(X = x)$$

$$E(X^2) = \sum_{x \in S} x^2 f(x) = \sum_{x \in S} x^2 P(X = x)$$

$$E(\sqrt{X}) = \sum_{x \in S} \sqrt{x}\, f(x) \equiv \sum_{x \in S} \sqrt{x}\, P(X = x)$$

e.g. For rolling a die:

T	1	2	3	4	5	6
$P(T=t) \equiv f(t)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

$$E(T) = \sum_{t \in \{1,2,\cdots,6\}} t\, f(t) = \sum_{\in\{1,2,3\cdots,6\}} t\, f(t) = (1)f(1) + (2)f(2) + (3)f(3)$$
$$+ (4)f(4) + (5)f(5) + (6)f(6)$$

$$= (1)(1/6) + (2)(1/6) + (3)(1/6) + (4)(1/6)$$
$$+ (5)(1/6) + (6)(1/6)$$

(Facto out $1/6$) $$= 1/6\,(1+2+3+4+5+6) = 21/6 = 3.5$$

$$E(\sqrt{T}) = \sum_{t \in \{1,2,\ldots,6\}} \sqrt{t}\, f(t) = (\sqrt{1}) f(1) + \sqrt{2} f(2) + \sqrt{3} f(3)$$

$$+ \sqrt{4} f(4 + \sqrt{5} f(5) + \sqrt{6} f(6)$$

$$= \sqrt{1}(1/6) + \sqrt{2}(1/6) + \sqrt{3}(1/6) + \sqrt{4}(1/6) + \sqrt{5}(1/6) + \sqrt{6}(1/6)$$

$$= 1/6(\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6})$$

$$E(T^2) = \sum_{t \in \{1,2,\ldots,6\}} t^2 f(t) = 1^2 f(1) + 2^2 f(2) + 3^2 f(3) + 4^2 f(4) + 5^2 f(5)$$

$$+ 6^2 f(6)$$

$$= 1 \cdot 1/6 + 4 \cdot 1/6 + 9 \cdot 1/6 + 16 \cdot 1/6 + 25 \cdot 1/6 + 36 \cdot 1/6$$

$$= 1/6(1+4+9+16+25+36) = 91/6 = 15.166$$

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Variance of a Discrete Random Variable

For a discrete random varable, X, which can take values $x_1, x_2, \ldots$ and has mean μ we define **the variance of X** as:

$$Var(X) = \sum_{i=1}^{n} (x_i - \mu)^2 \cdot f(x_i)$$

There are other usefule ways to write this, most importantly:

$$Var(X) = \sum_{i=1}^{n} x_i^2 \cdot f(x_i) - \mu^2$$

which is the same as

$$\mathrm{Var} X = \sum_x (x - \mathrm{E}X)^2 f(x) = \mathrm{E}(X^2) - (\mathrm{E}X)^2.$$

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Variance of a Discrete Random Variable

Example:

Suppose that the we roll a die and let T be the number of dots facing up. What is the variance of T?

T	1	2	3	4	5	6
$f(t)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$
$t\,f(t)$	$1/6$	$2/6$	$3/6$	$4/6$	$5/6$	$6/6$
$t^2 f(t)$	$1/6$	$4/6$	$9/6$	$16/6$	$25/6$	$36/6$

$$E(T) = \sum_{t\in\{1,\dots,6\}} t\,f(t) = 21/6 \quad , \quad E(T^2) = \sum_{t\in\{1,\dots,6\}} t^2 f(t) = 91/6$$

So, using the Formula of variance:

$$Var(T) = E(T^2) - [E(T)]^2$$

$$= 91/6 - [21/6]^2 = 2.91$$

Do Not make big mistakes:

① for any r.v (X) : $0 \leq P(X=x) \leq 1$ (always)

② for any r.v (X) : $0 \leq F(x) \leq 1$ (CDF)

③ Unlike probabilities & CDF, E(X) can take any value $\in (-\infty, +\infty)$
But $var(X) \geq 0$ (always) i.e $var(X) \in (0, +\infty)$

General Info

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Variance of a Discrete Random Variable

Example

Say we have a random variable Q with pmf:

q		f(q)	
1		0.34	
2		0.10	
3		0.22	
7		0.34	

Find the variance and standard deviation

$$var(Q) = E(Q^2) - [E(Q)]^2$$

So, we need to find $E(Q)$ & $E(Q^2)$

by formulas,

$$E(Q) = \sum_{q \in \{1,2,3,7\}} q\,f(q) = (1)f(1) + (2)f(2) + (3)f(3) + (7)f(7)$$

$$= (1)(.34) + (2)(.1) + (3)(.22) + (7)(.34)$$

$$= .34 + .2 + .66 + 2.38$$

$$= 3.58$$

$$E(Q^2) = \sum_{q \in \{1,2,3,7\}} q^2 f(q) = (1)^2 f(1) + (2)^2 f(2) + (3)^2 f(3) + (7)^2 f(7)$$

$$= (1)(0.34) + (4)(0.1) + (9)(0.22) + 49(0.34)$$

$$= 0.34 + 0.4 + 1.98 + 16.66$$

$$= 19.38$$

Now: $Var(Q) = E(Q^2) - [E(Q)]^2 = 19.38 - (3.58)^2$
(by formula)

$$= 19.38 - 12.8164 = 6.5636$$

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Summary

Discrete Random Variables

- Discrete RVs are RVs that will take one of a countable set of values.
- When working with a discrete random variable, it is common to need or use the RV's
  - probability distribution: the values the RV can take and their probabilities
  - probability function: a function where $f(x) = P(X = x)$
  - cumulative probability function: a function where $F(x) = P(X \leq x)$.
  - mean: a value for X defined by $EX = \sum_x x \cdot f(x)$
  - variance: a value for X defined by $VarX = \sum_x (x - \mu)^2 \cdot f(x)$

Your Turn:

Chapter 5 Handout 1