Common Continous Distibutions

Normal Distribution

Background

Terms and Use

Common Dists

Uniform

Exponential

Normal

The Normal distribution

We have already seen the normal distribution as a "bell shaped" distribution, but we can formalize this.

The **normal** or **Gaussian** (μ, σ^2) distribution is a continuous probability distribution with probability density function (pdf)

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}} \quad \text{for all } x \in \mathbb{R}$$

$x \in (-\infty, +\infty)$

for $\sigma > 0$.

We then show that by $X \sim \mathrm{N}(\mu, \sigma^2)$

The Normal distribution

A normal random variable is (often) a finite average of many repeated, independent, identical trials.

> Mean width of the next 50 hexamine pallets
>
> Mean height of 30 students
>
> Rotal % yield of the next 10 runs of a chemical process

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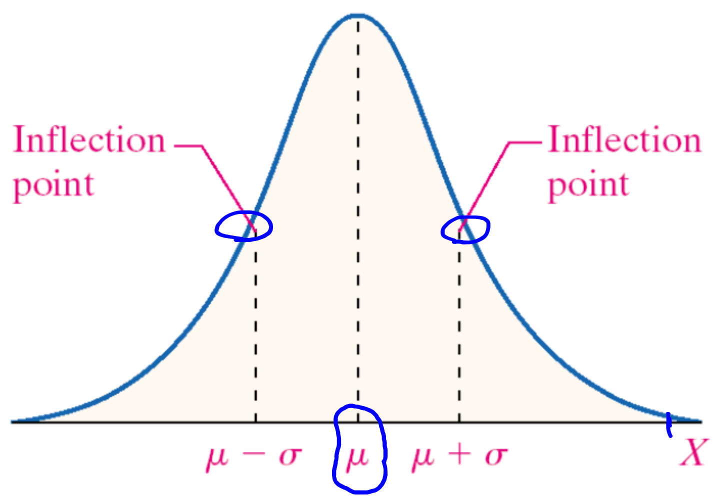
Normal Distribution's Center and Shape

Regardless of the values of μ and σ^2, the normal pdf has the following shape:

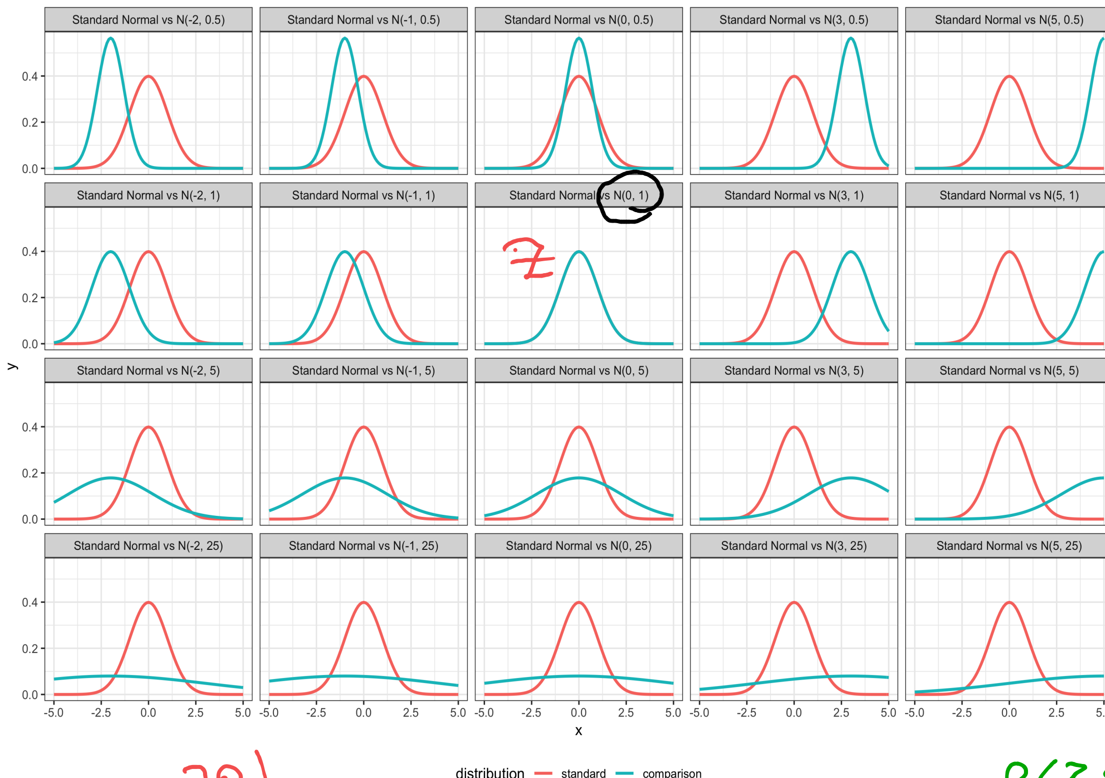


In other words, the distribution is centered around μ and has an inflection point at $\sigma = \sqrt{\sigma^2}$.

In this way, the value of μ determines the center of our distribution and the value of σ^2 deterimes the spread.

Normal Distribution's Center and Shape

Here we can see what differences in μ and σ^2 do to the shape of the shape of distribution



$P(X \geq 30)$

$X \sim N(3, 20) \rightsquigarrow Z \sim N(0,1) \Rightarrow P(Z \leq ?)$

and

Mean d~~na~~ Variance

of

Normal Distribution

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It is not obvious, but

- $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx =$ 1
- $\mathrm{E}X = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx =$ μ
- $\mathrm{Var}X = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx =$ σ^2

One poin~~e~~t before we go on

Standardization

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Definition

X: r.v but EX, V(X): not random (just values)

Standardization is the process of transforming a random variable, X, into the signed number of standard deviations by which it is is above its mean value.

$$Z = \frac{X - \mathrm{E}X}{\mathrm{SD}(X)}$$

Recall: $E(aX+b) = aEX + b$

$\boxed{Z \text{ has mean } 0}$

Recall $E(aX+b)$

Proof: $E(Z) = E\left(\frac{X - E(X)}{SD(X)}\right) = \frac{1}{SD(X)} E(X) - \frac{EX}{SD(X)}$

$= 0$

$\boxed{Z \text{ has variance (and standard deviation) } 1}$

Proof: $\mathrm{Var}\, Z = \mathrm{Var}\left(\frac{X}{SD(X)} - \frac{E(X)}{SD(X)}\right) = V\left(\frac{X}{SD(X)}\right) = \frac{1}{[SD(X)]^2} V(X) = 1$

$\to V(\cdot) = 0$

constant

$\mathrm{var}(aX+b) = a^2 V(X)$

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The Calculus I methods of evaluating integrals via anti-differentiation will fail when it comes to normal densities. They do not have anti-derivatives that are expressible in terms of elementary functions.

> This means we cannot find probabilities of a Normally distributed random variable by hand.
>
> So, what is the solution?
>
> Use computers or tables of values.

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The use of tables for evaluating normal probabilities depends on the following relationship. If $X \sim \text{Normal}(\mu, \sigma^2)$,

$$P[a \le X \le b] = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx$$

Standardization of $X \sim N(\mu, \sigma^2)$

$$= \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$E Z = 0$
$V(Z) = 1$

$$= P\left[\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right]$$

where $\boxed{Z \sim \text{Normal}(0, 1).}$

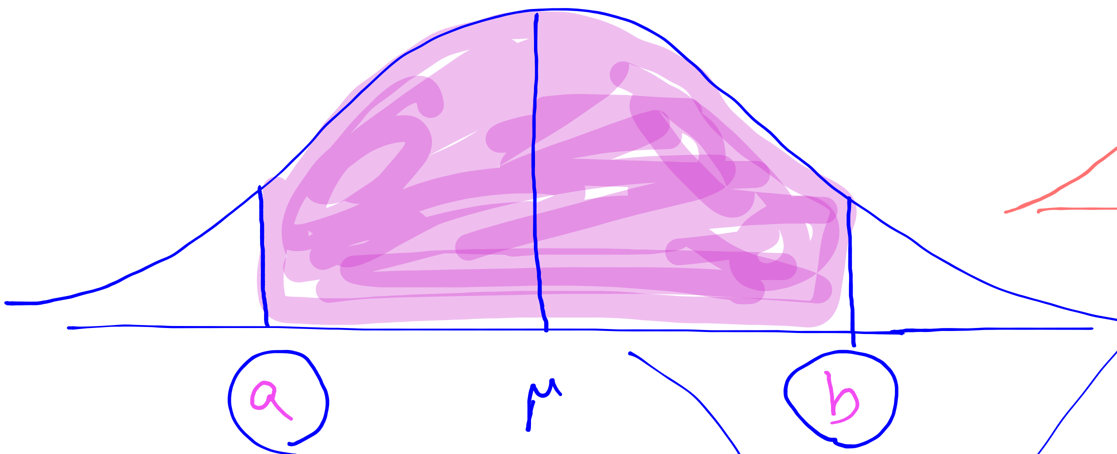
$X \sim N(\mu, \sigma^2)$

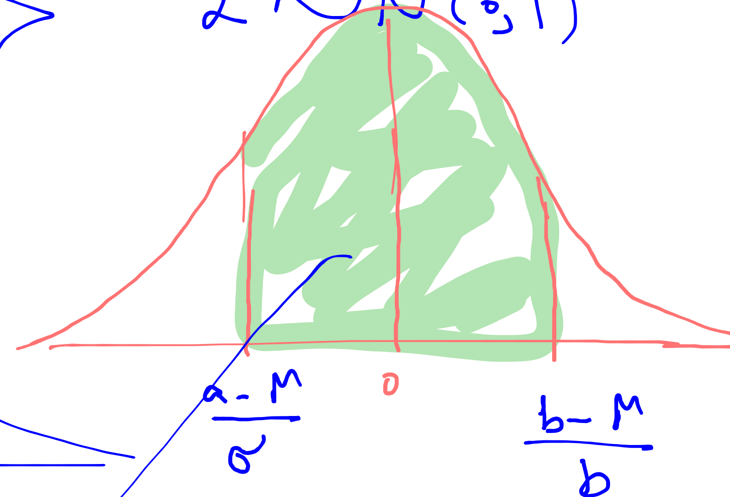
$P(a \leq X \leq b) = ?$

standardized

$\Longrightarrow$

$Z \sim N(0,1)$





$P\left(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right)$

equal area.

So, we can make any Normal(μ, σ^2) a standard Normal$(0,1)$!

$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{(x-\mu)^2}{-2\sigma^2}\right)$

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Standard Normal Distribution

The parameters are important in determining the probability, but because the pdf of a normal random variable is difficult to work with we often use the distribution with $\mu = 0$ and $\sigma^2 = 1$ as a reference point.

> **Definition: Standard Normal Distribution**
> The standard normal distribution is a normal distribution with $\mu = 0$ and $\sigma^2 = 1$. It has pdf
>
> $$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right)$$
>
> `

We say that a random variable is a "standard normal random variable" if it follows a standard normal distribution or that $Z \sim N(0, 1)$.

Standard Normal Distribution (cont)

There's no closed form CDF for Normal

It's worth pointing out the reason why the standard normal distribution is important. There is no "closed form" for the cdf of a normal distribution.

In other words, since we can't finish this step:

$$P(X \le x) = F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(t-\mu)^2} dt = ???$$

we have to estimate the value each time. However, we have already done this for *standard* normal random variables already in **Table B.3**

So if $Z \sim N(0,1)$ then $\boxed{P(Z \le 1.5)} = \boxed{F(1.5)} = 0.9332.$

The good news is that we can connect any normal probabilities to the values we have for the standard normal probabilities.

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Standard Normal Distribution (cont)

These facts drive the connection between different normal random variables:

> **Key Facts: Converting Normal Distributions**
>
> If $X \sim N(\mu, \sigma^2)$ and $Z = \dfrac{X - \mu}{\sigma}$ then $Z \sim N(0, 1)$
>
> If $Z \sim N(0, 1)$ and $X = \sigma Z + \mu$ then $X \sim N(\mu, \sigma^2)$

We use this connection as a way to avoid working with the normal pdf directly.

Standard Normal Distribution (cont)

A rule of thumb in dealing with questions about finding probabilities of Normally distributed probabilities of $N(\mu, \sigma^2)$:

> (1) Translate that question to standard Normal distribution. i.e. $Z \sim N(0, 1)$
>
> (2) Look it up in a table

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CDF of Standard Normal Distribution

The standard Normal distribution $ Z\sim N(0,1)$ plays an important rule in finding probabilities associated with a Normal random variable. The **CDF** of a standard Normal distribution is

$$\boxed{\Phi(z)} = F(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2} dt = P(Z \leq z).$$

Therefore, we can find probabilities for all normal distributions by tabulating probabilities for only the standard normal distribution. We will use a table of the **standard normal cumulative probability function**.

Recall: $X \sim N(\mu, \sigma^2) \Rightarrow Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

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Standard Normal Distribution (cont)

Example: Normal to Standard Normal

If $\boxed{X \sim N(3, 4)}$ then: $\sigma^2 = 4 \rightarrow SD(X) = \sqrt{4} = 2$

$Z = \frac{X-\mu}{\sigma}$

$$P(X \leq 6) = P\left(\frac{X-3}{2} \leq \frac{6-3}{2}\right)$$

$$= P(Z \leq 1.5) = F(1.5) = \Phi(1.5)$$

$$= 0.9332$$

where the valeu 0.9332 if found from **Table B.3**

Standard Normal Distribution (cont)

Example: Standard normal probabilities

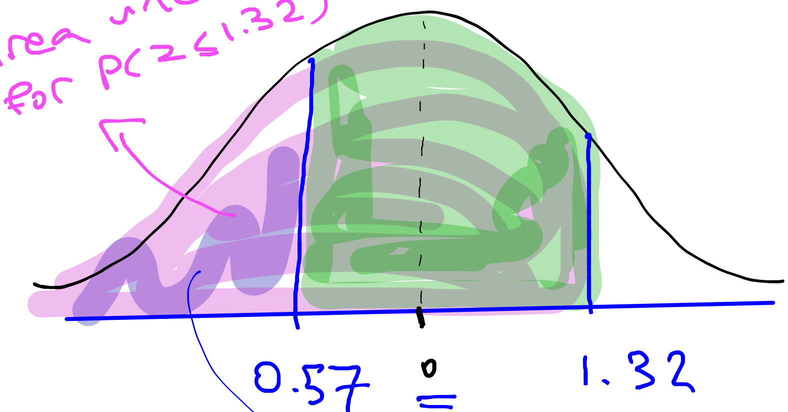
* $P[Z < 1.76]$ = 0.9608

Z ~ N(0,1)

the curve

Z ~ N(0,1)

area under
for P(Z ≤ 1.32)



$P[.57 < Z < 1.32]$

always useful to
draw a picture of
the std. Normal graph

= P(Z ≤ 1.32) − P(Z ≤ 0.57)

area under
the curve
for P(Z < 0.57)

= Φ(1.32) − Φ(0.57)

= 0.9066 − 0.7157

Example :

$$X \sim N(10, 49)$$

$$Z = \frac{X-\mu}{SD(X)} = \frac{X-\mu}{\sqrt{\sigma^2}}$$

① $P(X \geq 24) = P(X - 10 \geq 24 - 10)$

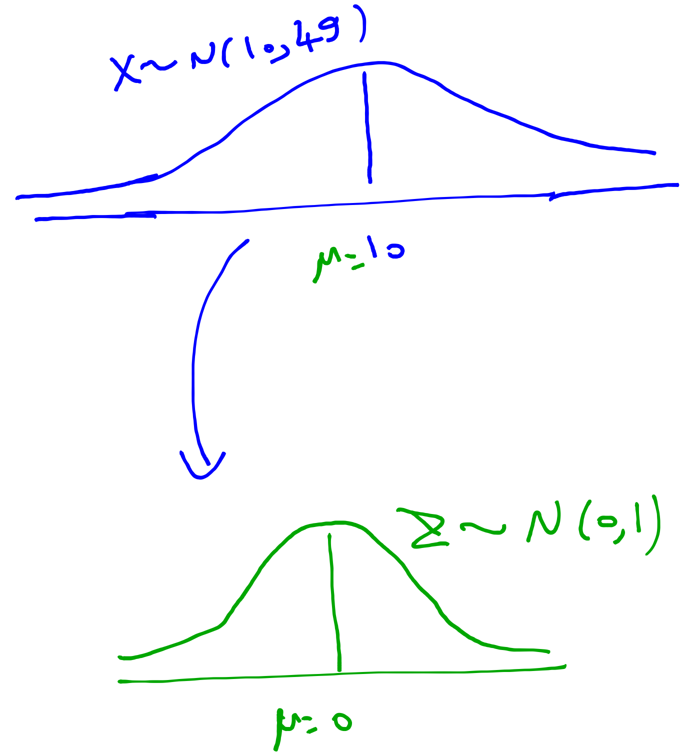
$= P\left(\frac{X-10}{7} \geq \frac{24-10}{7}\right)$

$= P(Z \geq 2)$

$= 1 - P(Z < 2)$

$= 1 - \Phi(2)$

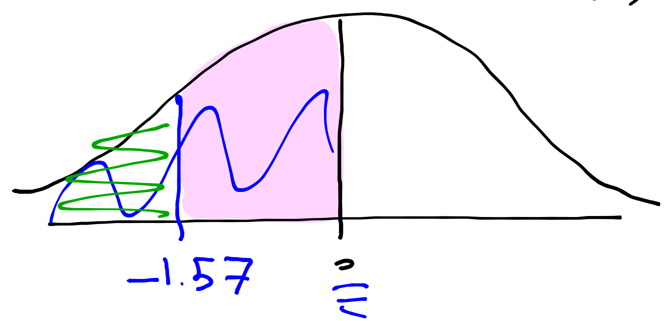
$= 1 - 0.9772$



$X \sim N(10, 49)$

$$P(-1 \le X \le 10) = P\left(\frac{-1-10}{7} \le \frac{X-10}{7} \le \frac{10-10}{7}\right)$$

$Z \sim N(0,1)$



$$= P\left(\frac{-11}{7} \le Z \le 0\right)$$

$$= P(-1.57 \le Z \le 0)$$

$$= P(Z \le 0) - P(Z \le -1.57)$$

$$= 0.5 - 0.0582$$

$$P(X \ge 2) = P\left(\frac{X-10}{7} \ge \frac{2-10}{7}\right) = P\left(Z \ge -\frac{8}{7}\right)$$

$$= 1 - P(Z \le -8/7) = 1 - \Phi(-8/7)$$

$P(Z \leq 8)$ ↙ 1.5

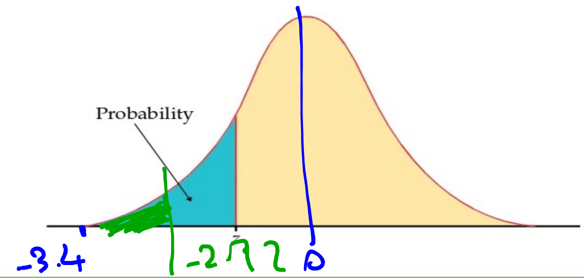


Table entry for z is the area under the standard normal curve to the left of z.

$P(Z \leq -2.72)$
$= 0.0033$

TABLE A

Standard normal probabilities

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

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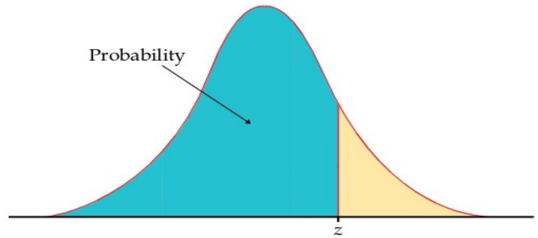

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TABLE A

Standard normal probabilities (continued)

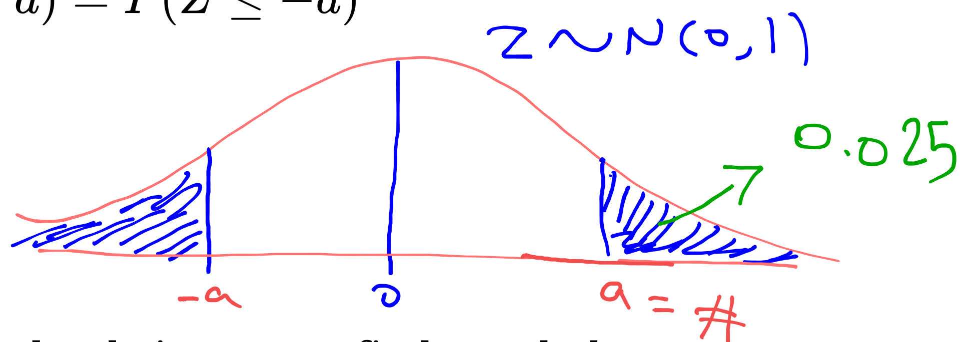
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(1.5)$

#

Some useful tips about standard Normal distribution

> By symmetry of the standard Normal distribution around zero
> $P(Z \geq a) = P(Z \leq -a)$



> We can also do it reverse, find z such that
> $P(-z \leq Z \leq z) = 0.95$
>
> $P(Z \geq \#) = 0.025$

① $P(Z \leq -\#) = 0.025$

by the table, $-\# = -1.96$

$\Rightarrow \boxed{\# = 1.96}$

② $P(Z \geq \#) = 0.025$

$\underbrace{1 - P(Z \leq \#)}$ $= 0.025$

$P(Z \leq \#) = 1 - 0.025$

$\phi(\#) = 0.975$

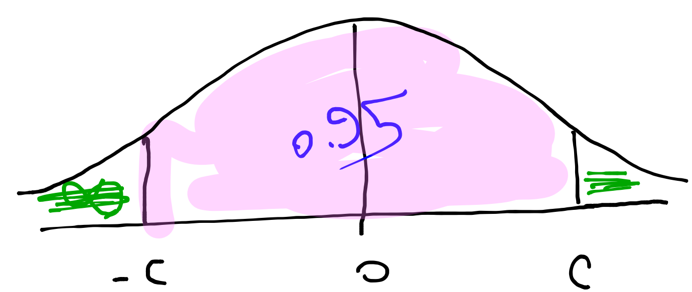
$\Rightarrow \underbrace{\# = 1.96}$

Example:

$$P(|Z| \le c) = 0.95$$

$$= \underbrace{P(-c < Z < c)}_{\|} = 0.95$$

$Z \sim N(0,1)$



$= 1 - 2P(Z > c)$

or

$= 1 - 2P(Z < -c)$

$$1 - 2P(Z < -c) = 0.95$$

$$\Rightarrow 1 - 2\Phi(-c) = 0.95$$

$$\Rightarrow 2\Phi(-c) = 1 - 0.95$$

$$\Rightarrow 2\Phi(-c) = 0.05$$

$$\Rightarrow \Phi(-c) = \frac{0.05}{2} = 0.025$$

by the table : $-c = -1.96$

$$\boxed{\Rightarrow c = 1.96}$$