Continuous Random Variables

Terminology, Use, and Common Distributions

What is a Continuous Random Variable?

Background on Continuous Random Variable

Along with discrete random variables, we have continuous random variables. While discrete random variables take one specific values from a *discrete* (aka countable) set of possible real-number values, continous random variables take values over intervals of real numbers.

> **def: Continuous random variable**
> A continuous random variable is a random variable which takes values on a continuous interval of real numbers.

The reason we treat them differently has mainly to do with the differences in how the math behaves: now that we are dealing with interval ranges, we change summations to integrals.

$P(X \in (-2, +2)) =$ Discrete: $P(-1 \leq X \leq 1) = \sum_{X \in \{-1, 0, 1\}} f(x)$

cont.: $P(-2 \leq X \leq 2) = \int_{-2}^{2} f(x)\, dx$

Background

Examples of continuous random variable:

What?

Z is the amount of torque required to lossen the next bold (not rounded)

t

2, 2.1, 2.001, 1.999

T is the time you will wait for the next bus

C is the outside temprature at 11:49 pm tomorrow

$T \in (-50, 120\,^{\circ}F)$

L is the length of the next manufactured metal bar

V is the yield of the next run of process

99.6%

99.7%

Terminology and Usage

Probability Density Function

Terminology

pdf

Since we are now taking values over an interval, we can not $\boxed{\text{"add up"}}$ probabilities with our probability function anymore. Instead, we need a new function to describe probability:

> **def: probability density function**
> A probability density function (pdf) defines the way the probability of a continuous random variable is distributed across the interval of values it can take. Since it represents probability, the probability function must always be non-negative. Regions of higher density have higher probability.

In discrete r.v : Probability Mass Function (PMF).

Probability Density Function

Validity of a *pdf*

Any function that satisfies the following can be a probability density function:

the highest possible value for X.

support of X

1. $\int_{-\infty}^{\infty} f(x)dx = 1$

the lowest possible value for r.v X

2. $f(x) \geq 0$ for all x in $(-\infty, \infty)$

and such that for all $a \leq b$,

$$P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b) = P(a < X < b) = \int_a^b f(x)dx.$$

Recall: In discrete r.v:

$$\sum_{x \in S_x} f(x) = 1$$

e.g: $P(-\infty < Z < 2) = \int_{-\infty}^{2} f(z)\, dz$

Background

Terms and Use

pdf

Probability Density Function

With continuous random variables, we use pdfs to get probabilities as follows:

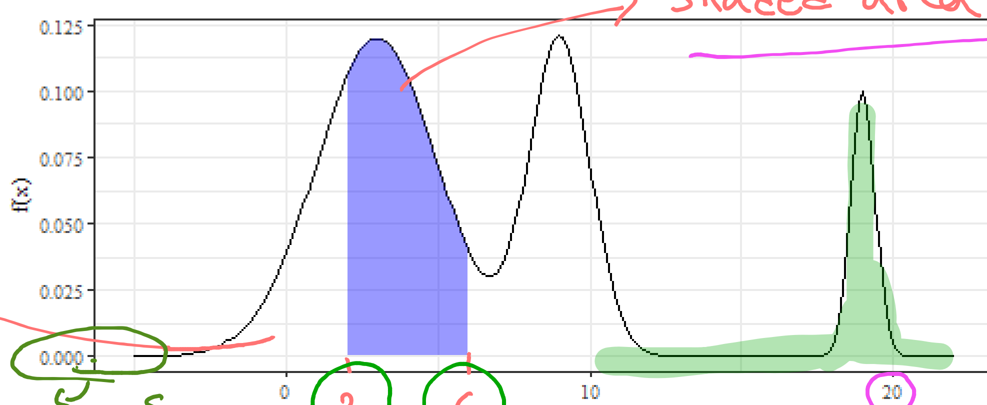
> For a continuous random variable X with probability density function $f(x)$,
>
> $$P(a \leq X \leq b) = \int_a^b f(x)dx$$
>
> `
>
> for any real values a, b such that $\boxed{a \leq b}$

a generic pdf:

total area under the curve: $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

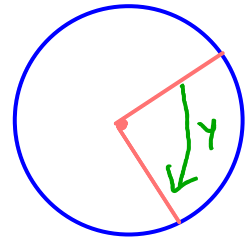
$f(x) \geq 0 \;\; \forall x \in S_x$

shaded area: $P(2 \leq X \leq 6)$



$\int_0^{20} f(x)\,dx = 1$

Example



Consider a de-magnetized compass needle mounted at its center so that it can spin freely. It is spun clockwise and when it comes to rest the angle, θ, from the vertical, is measured. Let

$$Y = \text{the angle measured after each spin in radians}$$

What values can Y take?

$0 \le Y \le 2\pi$

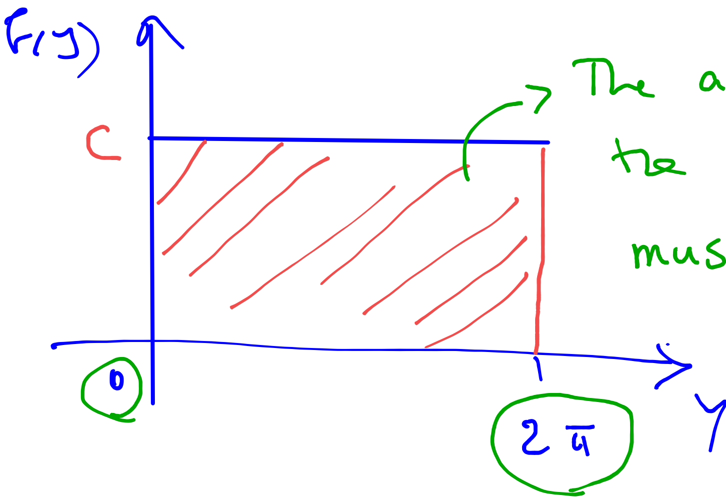
What form makes sense for $f(y)$?

$$f(y) = \begin{cases} C & 0 \le Y \le 2\pi \\ 0 & \text{o.w.} \end{cases}$$

y has a positive probability between $(0, 2\pi)$,

and it is equally likely to land at any angle.

(because it can spin freely)



The area under the curve (rectangular area) must be 1.

Example

If this form is adopted, that what must the pdf be?

$$\int_0^{2\pi} \underbrace{\boxed{f(x)}}_{c}\, dx = 1 \Rightarrow \int_0^{2\pi} c\, dx = 1$$

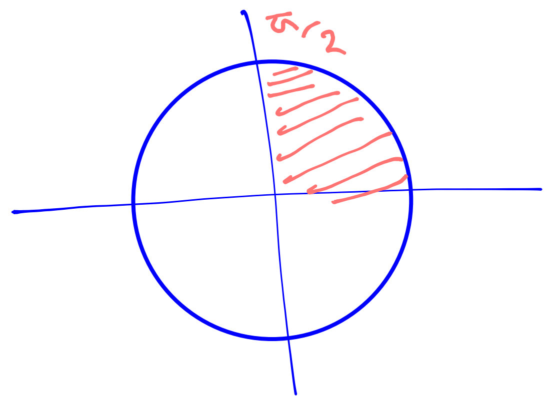
$$\Rightarrow c \int_0^{2\pi} dx = 1$$

$$\Rightarrow c\left(x \Big|_0^{2\pi}\right) = 1 \Rightarrow c(2\pi) = 1 \Rightarrow c = 1/2\pi$$

$$f(x) = \begin{cases} 1/2\pi & 0 \le y \le 2\pi \\ 0 & \text{o.w} \end{cases}$$

Using this pdf, calculate the following probabilities:

- $P[Y < \frac{\pi}{2}]$



$$\Rightarrow P(Y < \pi/2) = \int_0^{\pi/2} f(y)\, dy$$

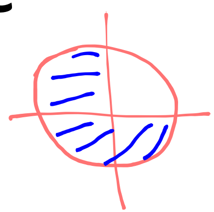
$$= \int_0^{\pi/2} 1/2\pi \, dy = \frac{1}{2\pi}\int_0^{\pi/2} dy$$

$$= \frac{1}{2\pi}\left(y \Big|_0^{\pi/2}\right) = \frac{1}{2\pi}\left(\pi/2 - 0\right) = \frac{\pi}{4\pi} = 1/4$$

Example

Terms and Use

pdf



- $P[\frac{\pi}{2} < Y < 2\pi] = \int_{\pi/2}^{2\pi} f(y)\,dy = \int_{\pi/2}^{2\pi} \frac{1}{2\pi}\,dy$

$= \frac{1}{2\pi}\int_{\pi/2}^{2\pi} dy = \frac{1}{2\pi}\left(y\Big|_{\pi/2}^{2\pi}\right)$

$= \frac{1}{2\pi}\left(2\pi - \frac{\pi}{2}\right) = 1 - \frac{1}{4} = \frac{3}{4}$

$P(\pi/2 < Y < 2\pi)$

$= 1 - P(0 \leq Y \leq \pi/2)$

- $P[Y = \frac{\pi}{6}] = 0$

$P(\pi/6 \leq Y \leq \pi/6) = \int_{\pi/6}^{\pi/6} f(y)\,dy = \int_{\pi/6}^{\pi/6} \frac{1}{2\pi}\,dy$

$= \frac{1}{2\pi}\left(\int_{\pi/6}^{\pi/6} dy\right) = \frac{1}{2\pi}\left(\frac{\pi}{6} - \frac{\pi}{6}\right) = 0$

Background

Terms and Use

pdf

cdf

Cumulative Density Function (CDF)

We also have the cumulative density function for continuous random variables:

> **def: Cumulative density function (cdf)** For a continous random variable, X, with pdf f(x) the cumulative density function $F(x)$ is defined as the probability that X takes a value less than or equal to x which is to say
>
> $$\underline{F(x) = P(X \leq x)} = \int_{-\infty}^{x} f(t)dt$$

TRUE FACT: the Fundamental Theorem of Calculus applies here:

$$\boxed{\frac{d}{dx}F(x) = f(x)}$$

Recall: in discrete r.v:

$$F(x) = P(X \leq x) = \sum_{x=0}^{x} f(x)$$

Background

Terms and Use

pdf

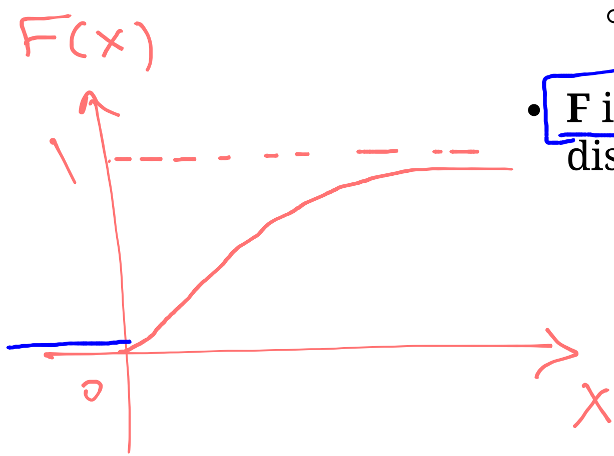
cdf

Cumulative Density Function (**CDF**)

Properties of CDF for continuous random variables

As with discrete random variables, F has the following properties:

- **F** is monotonically increasing (i.e it is never decreasing)
- $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$
    - This means that $0 \leq F(x) \leq 1$ for **any CDF**
- **F** is *continuous.* (instead of just right continuous in discrete form)



Example: ① For the following pdf, find the CDF:

$$f_X(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & o.w \end{cases} \longrightarrow F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} f(t)\,dt$$

$$= \int_0^x e^{-t}\,dt = \left(-e^{-t}\Big|_0^x\right) = 1 - e^{-x}, \quad x > 0$$

② The CDF of r.v Y is given as $F_X(x) = \begin{cases} 1 - e^{-2y} & y > 0 \\ 0 & o.w \end{cases}$

what is the pdf of Y?

$$f(y) = \frac{d}{dy} F(y) = \frac{d}{dy}\left(1 - e^{-2y}\right) = \begin{cases} 2e^{-2y} & y > 0 \\ 0 & o.w \end{cases}$$

Mean and Variance

of

Continuous Random Variables

Background

Terms and Use

pdf

cdf

E(X), V(X)

Expected Value and Variance

Expected Value

As with discrete random variables, continuous random variables have expected values and variances:

> **def: Expected Value of Continuous Random Variable**
> For a continous random variable, X, with pdf f(x) the expected value (also known as the mean) is defined as
>
> $$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

We often use the symbol μ for the mean of a random variable, since writing $E(X)$ can get confusing when lots of other parenthesis are around. We also sometimes write EX.

e.g.: $E(\sqrt{x}) = \int_{-\infty}^{+\infty} \sqrt{x}\, f(x)\, dx$

Background

Terms and Use

pdf

cdf

E(X), V(X)

Expected Value and Variance

Variance

def: Variance of Continuous Random Variable
For a continous random variable, X, with pdf f(x) and expected value μ, the variance is defined as

$$E(X-\underbrace{E(x)}_{\mu})^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x) dx$$

which is identical to saying

$$\boxed{V(X) = E(X^2) - E(X)^2}$$

We will sometimes use the symbol σ^2 to refer to the variance and you may see the notation $VarX$ or VX as well.

Background

Terms and Use

pdf

cdf

E(X), V(X)

Expected Value and Variance

Sdandard Deviation (SD)

We can also use the variance to get the standard deviation of the random variable:

> **def: Standard Deviation of Continuous Random Variable**
> For a continous random variable, X, with pdf f(x) and expected value μ, the standard deviation is defined as:
>
> $$\sigma = \sqrt{\sigma^2} = \sqrt{\int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)dx}$$

Background

Terms and Use

pdf

Expected Value and Variance: Example

Library books

Let X denote the amount of time for which a book on 2-hour hold reserve at a college library is checked out by a randomly selected student and suppose its density function is

$$f(x) = \begin{cases} 0.5x & \boxed{0 \le x \le 2} \\ 0 & \text{otherwise} \end{cases}$$

Calculate $\mathrm{E}X$ and $\mathrm{Var}X$.

by def. : $\boxed{EX} = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^2 x \cdot \frac{x}{2}\, dx = \frac{1}{2}\int_0^2 x^2\, dx$

$= \frac{1}{2}\left(\frac{x^3}{3}\Big|_0^2\right) = \frac{1}{2}\left(\frac{2^3}{3} - 0\right)$

$= \frac{8}{2 \times 3} = \frac{8}{6} = 1.333$

by def. : $Var(x) = E x^2 - [E(x)]^2$

$$E x^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^2 x^2 \cdot \boxed{\frac{x}{2}}\, dx = \int_0^2 \frac{x^3}{2}\, dx$$

$$= \frac{1}{2} \int_0^2 x^3\, dx = \frac{1}{2} \left(\frac{x^4}{4} \Big|_0^2 \right)$$

$$= \frac{1}{2} \left(\frac{2^4}{4} - 0 \right) = \frac{16}{8} = 2$$

$$Var\, X = E x^2 - (E X)^2 = 2 - \left(8/6 \right)^2$$

$$= 2/9$$

$$\Rightarrow SD(x) = \sqrt{Var\, x} = \sqrt{2/9} = \sqrt{2}/3$$

An important point about Expected Value and Variance of Random Variables

Background

Expected Value and Variance:

Terms and Use

pdf

For a linear function, $g(X) = aX + b$, where a and b are constants, $g(x)$

> $$\mathrm{E}(aX + b) = \boxed{a\mathrm{E}(X) + b}$$
>
> $$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$$

(my point: E(constants) = constants. E(2)=2

$E(x) = n.p$

$V(x) = n.p(1-p)$

var(constants) = 0

(eg var(2.3) = 0

e.g Let $X \sim Binomial(5, 0.2)$. What is the expected value and variance of 4X- 3?

x indep.

example: $\mathrm{var}(\sqrt{2}X + \frac{X^2}{2}) = \mathrm{Var}(\sqrt{2}X) + V(X^2/2)$

$= (\sqrt{2})^2 V(X) + \left(\frac{1}{2}\right)^2 V(X^2)$

$E(4X-3) = 4(EX) - 3 = 4(1) - 3 = 1$

$EX = 5(0.2) = 1$

$\mathrm{Var}(4X-3) = 4^2\,\mathrm{Var}(X) - 0 = 16(5(0.2)(1-0.2)) = 16.(0.8)$

$$E(g(x)) = E(aX+b)$$

$$= E(aX) + E(b)$$

The expected value of constants are constants.

a, b are constants

$$= a\,E(X) + b$$

e.g $X \sim f(x)$, $\underline{x > 0}$

$$E(aX+b) = \int_0^\infty (ax+b) f(x)\,dx = \int_0^\infty ax\,f(x)\,dx$$

$$+ \int_0^\infty b\,f(x)\,dx = a\underbrace{\int_0^\infty x\,f(x)\,dx}_{} + b\underbrace{\int_0^\infty f(x)\,dx}_{}$$

$$= aE(X) + b$$

$$Var(g(x)) = Var(aX + b)$$

$$= var(aX) + \cancelto{0}{var(b)}$$

$$= a^2 var(X)$$

$$= a^2 \sigma_X^2$$

Common Distributions

Uniform Distribution

Background

Terms and Use

Common Dists

Uniform

skip

Common continuous Distributions

Uniform Distribution

For cases where we only know/believe/assume that a value will be between two numbers but know/believe/assume *nothing* else.

Origin: We know a the random variable will take a value inside a certain range, but we don't have any belief that one part of that range is more likely than another part of that range.

> **Definition: Uniform random variable**
> The random variable U is a uniform random variable on the interval $[a, b]$ if it's density is constant on $[a, b]$ and the probability it takes a value outside $[a, b]$ is 0. We say that U follows a uniform distribution or $U \sim uniform(a, b)$.

$a \leq b$

Uniform Distribution

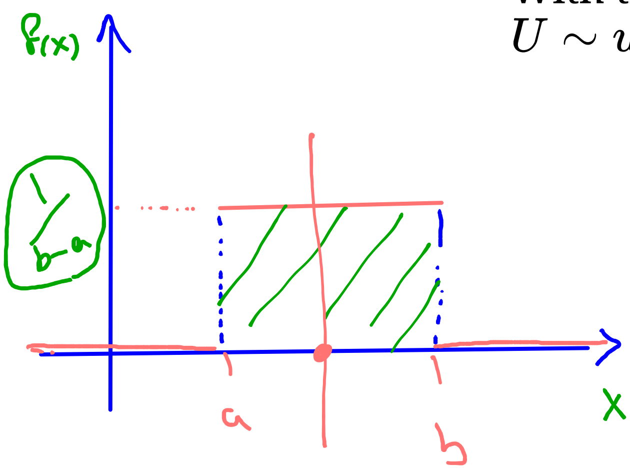
Definition: Uniform pdf
If U is a uniform random variable on $[a, b]$ then the probability density function of U is given by

$$f(u) = \begin{cases} \boxed{\frac{1}{b-a}} & a \leq u \leq b \\ 0 & o.w. \end{cases}$$

With this, we can find the for any value of a and b, if $U \sim uniform(a, b)$ the mean and variance are:

$$E(U) = \frac{1}{2}(b + a)$$

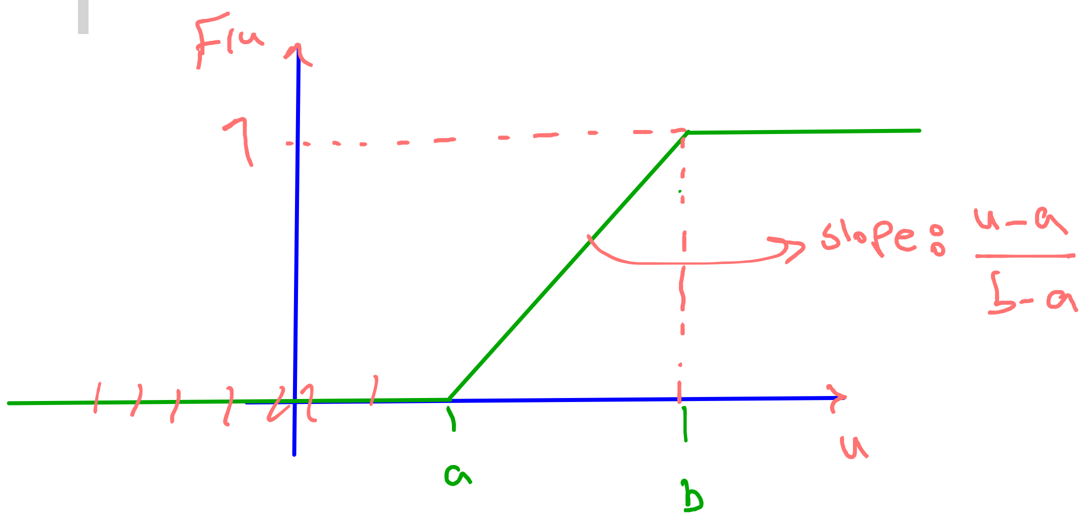
$$Var(U) = \frac{1}{12}(b - a)^2$$



Uniform Distribution

Definition: Uniform cdf
If U is a uniform random variable on $[a, b]$ then the cumulative density function of U is given by

$$F(u) = \begin{cases} 0 & u < a \\ \frac{u-a}{b-a} & a \leq u \leq b \\ 1 & u > b \end{cases}$$



Background

Terms and Use

Common Dists

Uniform

Uniform Distribution

A few useful notes:

- The most commonly used uniform random variable is $U \sim Uniform(0, 1)$.
- Again, this is useful if we want to use a random variable that takes values within an interval, but we don't think it is likely to be in any certain region.
- The values a and b used to determine the range in which $f(u)$ is not 0 are parameters of the distribution.

Common Continuous Distributions

Exponential Distribution

Background

Terms and Use

Common Dists

Uniform

Exponential

Exponential Distribution

Definition: Exponential random variable

Parameter.

An $\text{Exp}(\alpha)$ random variable measures the waiting time until a specific event that has an equal chance of happening at any point in time. (it can be cosidered the continous version of geometric distribution)

> Examples:
>
> - Time between your arrival at the bus station and the moment that bus arrives
> - Time until the next person walks inside the park's library
> - The time (in hours) until a light bulb burns out.

Background

Terms and Use

Common Dists

Uniform

Exponential

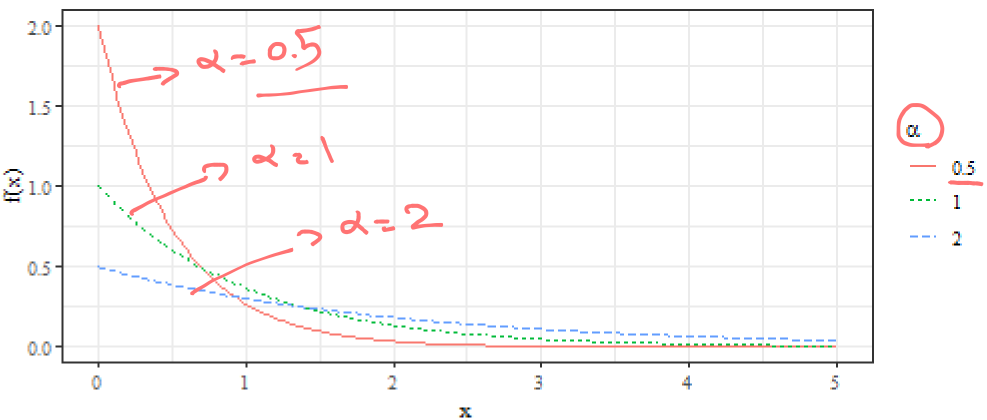
Exponential Distribution

Definition: Exponential pdf
If X is an exponential random variable with rate $\frac{1}{\alpha}$ then the probability density function of X is given by

$$f(\cancel{x}) = \begin{cases} \frac{1}{\alpha} e^{-\frac{x}{\alpha}} & x \geq 0 \\ 0 & o.w. \end{cases}$$

`

$\frac{1}{\alpha} e^{-x/\alpha}$



Background

Terms and Use

Common Dists

Uniform

Exponential

Exponential Distribution

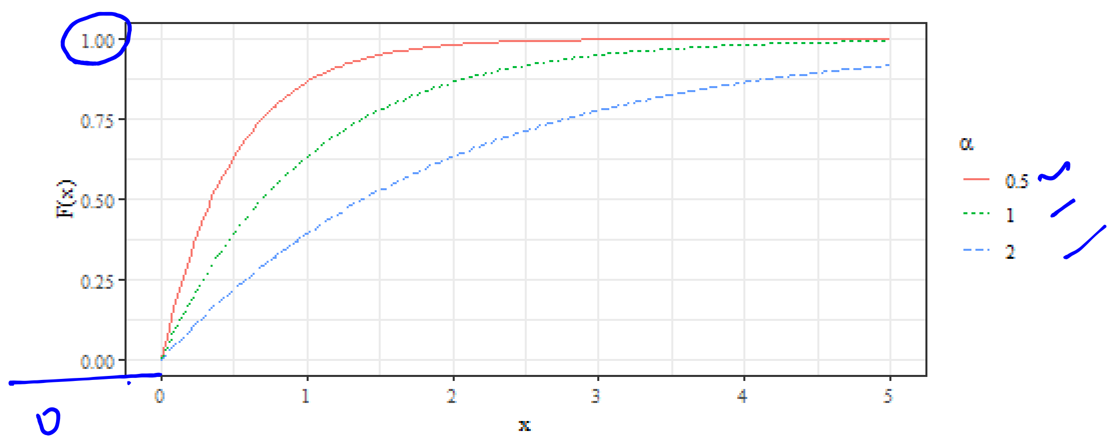
Close CDF

(Remember in discrete pmf Geometric has close CDF)

Definition: Exponential CDF
If X is a exponential random variable with rate $1/\alpha$ then the cumulative density function of X is given by

$$F(x) = \begin{cases} \boxed{1 - exp(-x/\alpha)} & 0 \leq x \\ 0 & x < 0 \end{cases}$$

Geom : $F(x) = 1 - (1-p)^x$



Mean and Variance of Exponential Distribution

Exponential Distribution

Definition: Exponential pdf
If X is an exponential random variable with rate $\frac{1}{\alpha}$ then the probability density function of X is given by

$$f(\cancel{x}) = \begin{cases} \frac{1}{\alpha} e^{-\frac{x}{\alpha}} & x \geq 0 \\ 0 & o.w. \end{cases}$$

` From this, we can derive:

$$\boxed{E(X) = \alpha}$$

$$\boxed{Var(X) = \alpha^2}$$

Exponential Distribution

Example: Library arrivals, cont'd

Recall the example the arrival rate of students at Parks library between 12:00 and 12:10pm early in the week to be about 12.5 students per minute. That translates to a $1/12.5 = .08$ minute average waiting time between student arrivals.

α

Consider observing the entrance to Parks library at exactly noon next Tuesday and define the random variable

$$T: \text{the waiting time (min) until the first student passes through the door.}$$

Using $T \sim \text{Exp}(.08)$, what is the probability of waiting more than 10 seconds (1/6 min) for the first arrival?

$\to \alpha$

$P(T > 10'') = P(T > 1/6) = 1 - P(T \le 1/6) = 1 - F_T(1/6)$

$f_T(t) = \begin{cases} \frac{1}{0.08} e^{-t/0.08} & t \ge 0 \\ 0 & \text{o.w.} \end{cases}$

$= 1 - \left[1 - e^{-\frac{1/6}{0.08}}\right] = e^{-\frac{1}{6(.08)}}$

Background

Terms and Use

Common Dists

Uniform

Exponential

Exponential Distribution

Example: Library arrivals, cont'd

T : the waiting time (min) until the first student passes through the door.

What is the probability of waiting less than 5 seconds?

(5 seconds $= \frac{1}{12}$ minute)

$P(T < 5 \text{ seconds}) = P(T < \frac{1}{12})$

$= P(T \leq \frac{1}{12})$

$= F_T(\frac{1}{12}) = 1 - \exp\left(-\frac{\frac{1}{12}}{0.08}\right)$

$\approx .6471$ ✓

1 α

5 60