STAT 305: Chapter 6

Introduction to formal statistical inference

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Chapter 6.1: Large-sample confidence intervals for a mean

Large Sample Confidence Interval

Formal statistical inference uses probability theory to quantify the reliability of data-based conclusions. We want information on a population.e.g

> - true mean fill weight of food jams
> - true mean strength of metal bars
> - true mean of the number of accidents on a highway in Iowa

We can then use:

1. Point estimates:

   e.g **sample mean** $\overline{X}$ of the strength of metal bars is 4.83.

   > We would then say that $\overline{X}$ is an estimate for true (population) mean μ.

Large Sample Confidence Interval

1. Interval estimates:
   μ is likely to be inside an interval. (e.g
   $\mu \in (2.84, 5.35)$)

Then we can say **we are confident that the true mean of the strength of metal bars (μ) is somewhere in the** $(2.84, 5.35)$

But the question is *how confident*?

Large Sample Inference

Large Sample Confidence Interval

Many important engineering applications of statistics fit the following mold. Values for parameters of a data-generating process are unknown. Based on data, the goal is

> 1.identify an inteval of values likely to contain an *unknown parameter*
>
> 2.qualify "how likely" the interval is to cover the correct value of the unknown parameter.

Confidence Interval

Definition and the use

Confidence Interval

Definition: confidence interval for a *parameter* (or function of one or more parameters) is a *data-based interval* of numbers thought likely to contain the parameter (or function of one or more parameters) possessing a stated probability-based confidence or reliability.

> A confidence interval is a realization of a **random interval**, an interval on the real line with a random variable at one or both of the endpoints.

$$P\left(\bar{x} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{1-\alpha/2}\sigma/\sqrt{n}\right) = 1-\alpha$$

$$P(5.5'' < \mu \leq 6.3'') = 0.95$$

Example:[Instrumental drift]

Let Z be a measure of instrumental drift of a random voltmeter that comes out of a certain factory. Say $Z \sim N(0, 1)$. Define a random interval:

$$(Z - 2, Z + 2)$$

What is the probability that -1 is inside the interval?

$$\begin{aligned} P(-1 \text{ is in } (Z-2, Z+2)) &= P(Z - 2 < -1 < Z + 2) \\ &= P(Z - 1 < -1 < Z + 3) \\ &= P(-1 < -Z < 3) \\ &= P(-3 < Z < 1) \\ &= \Phi(1) - \Phi(-3) \\ &= 0.84. \end{aligned}$$

with 84% Confidence, -1 Falls within the interval

$(Z-2, Z+2)$ when $Z \sim N(0,1)$

Large Sample Inference

Confidence Interval

Example:[More practice]

Calculate:

- $P(2 \text{ in } (X-1, X+1)), X \sim N(2,4)$

μ

$\sigma^2 = 4$

$$
\begin{aligned}
P(\boxed{2} \in (X-1, X+1)) &= P(X-1 < 2 < X+1) \\
&= P(-1 < 2 - X < 1) \\
&= P(-1/2 < \cancel{X} < 1/2) \\
&= \Phi(1/2) - \Phi(-1/2) \\
&= 0.6915 - 0.3085 \\
&= 0.383
\end{aligned}
$$

$P(-1-2 < 2-X < 1-2)$

$= P(-3 < -X < -1)$

$= P(3 > X > 1) = P(1 < X < 3)$

$= P\left(\frac{1-2}{2} < \frac{X-2}{2} < \frac{3-2}{2}\right) = P(-1/2 < Z < +1/2)$

with 38% confidence, 2 falls within the interval $(X-1, X+1)$ when $X \sim N(2, 4)$

Example:[Abstract random intervals]

Let's say $X_1, X_2, \ldots, X_n$ are iid with $\boxed{n \geq 25,}$ mean μ, variance σ^2. We can find a random interval that provides a lower bound for μ with $1-\alpha$ probability:

We want **A** such that $\boxed{P(\mu \in (A, +\infty)) = 1 - \alpha.}$
We know by CLT:

$$\overline{X} \approx N(\mu, \sigma^2/n)$$

Find A s.t

$P(\mu \in (A, +\infty)) = 1 - \alpha$

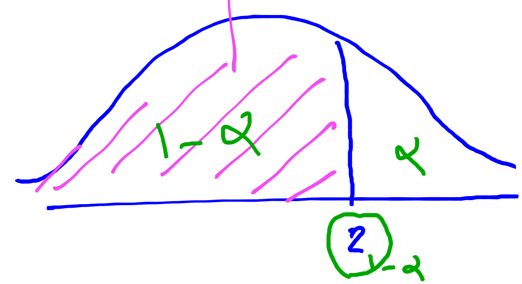
Therefore,

$$\rightarrow \text{(standardization)} \quad \underbrace{\frac{\overline{X} - \mu}{\sigma/\sqrt{n}}}_{Z} \sim N(0, 1).$$

$\Phi(z) = 1 - \alpha$

+

we know $\underbrace{P(Z \leq z)}_{\Phi(z)} \approx 1 - \alpha$



Large Sample Inference

Confidence Interval

Example:[Abstract random intervals]

Then

$$P(\overbrace{\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}}^{Z} \leq Z_{1-\alpha}) \approx 1-\alpha$$

solve it for μ

(Recall: we're looking for a CI for μ)

$$\Rightarrow P(\overline{X} - Z_{1-\alpha}\frac{\sigma}{\sqrt{n}} \leq \mu) \approx 1-\alpha$$

$$\Rightarrow P(\mu \in (\overline{X} - Z_{1-\alpha}\frac{\sigma}{\sqrt{n}}, +\infty)) \approx 1-\alpha$$

Now if we set

$$A = \boxed{\overline{X} - Z_{1-\alpha}\frac{\sigma}{\sqrt{n}}}$$

then we have

$$P(\mu \in (A, +\infty)) \approx 1-\alpha$$

Large Sample Inference

Confidence Interval

Example:[Abstract random intervals]
Calculate:

$$\boxed{P(\mu \in (\overline{X} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{X} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}))}, X \sim N(\mu, \sigma^2)$$

$$P(\mu \in (\overline{X} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{X} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}))$$

Solve it for μ

$$= P(\overline{X} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}})$$

$$= P(-z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu - \overline{X} < z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}})$$

$$= P(-z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} < \boxed{\overline{X} - \mu} < z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}})$$

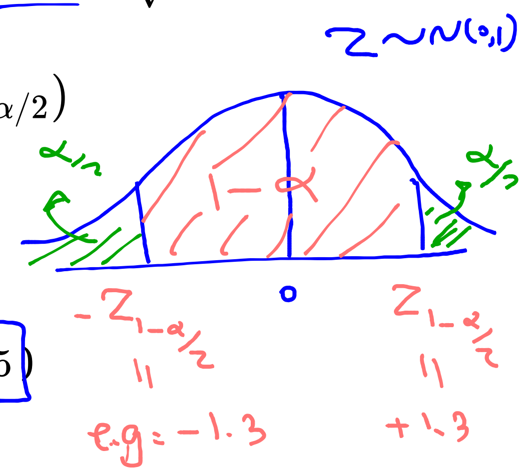
$$= P(-z_{1-\alpha/2} < \boxed{\frac{\overline{X} - \mu}{\sigma/\sqrt{n}}} < z_{1-\alpha/2})$$

$$= P(-z_{1-\alpha/2} < Z < z_{1-\alpha/2})$$

$$\approx 1 - \alpha$$

(μ is a parameter. It doesn't have any dist.
But $Z \sim N(0,1)$)

Recall: we find CI for unknown parameters of the population.



(The last result is by **CLT** assuming that $\boxed{n \geq 25}$)

Example:[Abstract random intervals]

So, μ falls within the interval

* $(\overline{X} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{X} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}})$ with the probability of $1 - \alpha$ for $X \sim N(\mu, \sigma^2)$

$\alpha = 0.05$

i.e we are 100(1-α)% (e.g 95%) confident that the true parameter μ in the dist. $X \sim N(\mu, \sigma^2)$ falls within the interval

$$\left(\bar{X} - Z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + Z_{1-\alpha/2}\,\sigma/\sqrt{n}\right)$$

A large-n confidence interval for μ involving σ

Large Sample Inference

Confidence Interval

CI for μ

A Large-n confidence interval for μ involving σ

100

A **(1- α)%** **confidence interval** for an unknown parameter is the realization of a random interval that contains that parameter with probability $1-\alpha$.

> α is called the **confidence level**

e.g: $\alpha = 0.02$

$\Rightarrow 100(1-0.02)\% = 98\%$

For random variables $X_1, X_2, \ldots, X_n$ iid with $E(X_1) = \mu$, $Var(X_1) = \sigma^2$, a $(1-\alpha)\%$ confidence interval for μ is

100

For Large n $(n \geq 25)$

$$\left(\overline{x} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

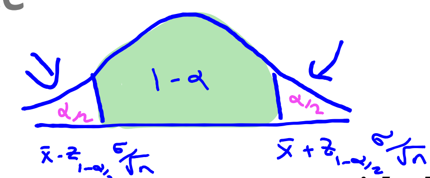
which is a **realization** from the random interval

$$\left(\overline{X} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{X} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right).$$

CI for μ

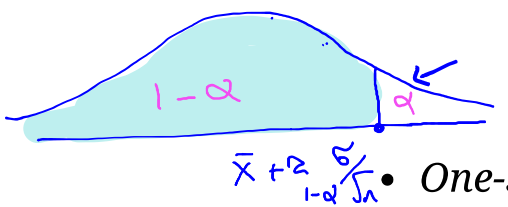
In General (for large n)

- *Two-sided* $100(1-\alpha)\%$ confidence interval for μ



$$\left(\bar{x} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

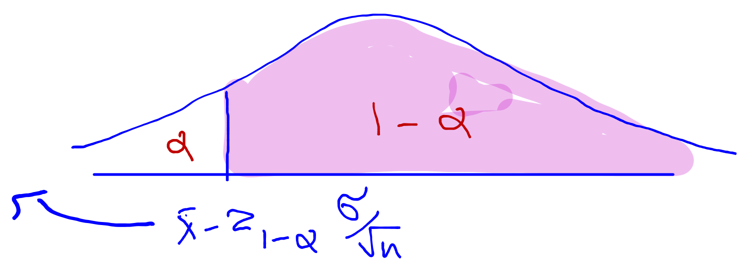
- *One-sided* $100(1-\alpha)\%$ confidence interval for μ with a upper confidence bound



$$\left(-\infty\ ,\ \bar{x} + z_{1-\alpha}\frac{\sigma}{\sqrt{n}}\right)$$

- *One-sided* $100(1-\alpha)\%$ confidence interval for μ with a lower confidence bound

$$\left(\bar{x} - z_{1-\alpha}\frac{\sigma}{\sqrt{n}}\ ,\ +\infty\right)$$



lower bound for μ

Large Sample Inference

Confidence Interval

CI for μ

$100(1-\alpha)\% = 90\%$

$\Rightarrow 1-\alpha = 0.9$

$\boxed{\alpha = 0.1}$

$Z_{0.95} = 1.64$

$Z_{0.9} = 1.96$ (check!)

Example:[Fill weight of jars]

Suppose a manufacturer fills jars of food using a stable filling process with a known standard deviation of $\sigma = 1.6$g. We take a sample of $n = 47$ jars and measure the sample mean weight $\overline{x} = 138.2$g. A two-sided 90% confidence interval, $\alpha = 0.1$, for the true mean weight μ is:

$$\left(\overline{x} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

$$= \left(\overline{x} - z_{1-0.1/2}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{1-0.1/2}\frac{\sigma}{\sqrt{n}}\right)$$

$$= \left(138.2 - z_{.95}\frac{1.6}{\sqrt{47}}, \ 138.2 + z_{.95}\frac{1.6}{\sqrt{47}}\right)$$

$$= (038.2 - 1.64(.23)\ , \ 138.2 + 1.64(.23))$$

$$= (137.82\ , \ 138.58)$$

0.38

or we can write it as 138.2 ± 0.38g

Large Sample Inference

Confidence Interval

CI for μ

Example:[Fill weight of jars]

* Interpretation: (μ)

> We are 90% **confident** that the **true mean** is between 137.82g and 138.58g
>
> or we can say
>
> If we took 100 more samples of 47 jams each, roughly *90 of those samples would have a confidence interval containing the true mean fill weight*

Large Sample Inference

Confidence Interval

CI for μ

Example:[Fill weight of jars]

What if we just want to be sure that the true mean fill weight is high enough?

We could use a one-sided 90% CI with a lower bound:

$\alpha = 0.1$

$$
\begin{aligned}
&\left(\bar{x} - \boxed{z_{1-\alpha}}\frac{\sigma}{\sqrt{n}}\ ,\ \ +\infty\right) \\
&= \left(138.2 - z_{.9}\frac{1.6}{\sqrt{47}}\ ,\ \ +\infty\right) \\
&= \left(137.91\ ,\ \ +\infty\right)
\end{aligned}
$$

Then we would say:

> We are 90% confident that the true mean fill weight is above 137.91

μ

Large Sample Inference

Confidence Interval

CI for μ

Example:[Hard disk failures]

F. Willett, in the article "The Case of the Derailed Disk Drives?" (**Mechanical Engineering**, 1988), discusses a study done to isolate the cause of link code A failure in a model of Winchester hard disk drive.

For each disk, the investigator measured the breakaway torque (in. oz.) required to loosen the drive's interrupter flag on the stepper motor shaft.

Breakaway torques for 26 disk drives were recorded, with a sample mean of 11.5 in. oz. Suppose you know the true standard deviation of the breakaway torques is 5.1 in. oz.

Large Sample Inference

Confidence Interval

CI for μ

Example:[Hard disk failures]

Calculate and interpret:

- A two-sided 90% confidence interval for the true mean breakaway torque of the relevant type of Winchester drive.

$100(1-\alpha)\% = 90\%$

$\Rightarrow 1-\alpha = 0.9$

$\boxed{\alpha = 0.1}$

$$
\begin{aligned}
&(\bar{x} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}) \\
&= (\bar{x} - z_{1-0.1/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-0.1/2}\frac{\sigma}{\sqrt{n}}) \\
&= (11.5 - z_{.95}\frac{5.1}{\sqrt{26}}, \ 11.5 + z_{.95}\frac{5.1}{\sqrt{26}}) \\
&= (11.5 - 1.64(1.0002) \ , \ 11.5 + 1.64(1.0002)) \\
&= (9.86 \ , \ 13.14)
\end{aligned}
$$

Interpretation: we are 90% confident that the true mean breaking torque lies between 9.86 and 13.14 in.oz.

scale

Large Sample Inference

Confidence Interval

CI for μ

Example: [Width of a CI]

If you want to estimate the breakaway torque with a 2-sided, $\boxed{95\%}$ confidence interval with ± 2.0 in. oz. of precision, what sample size would you need?

> $\boxed{\text{Interval precision = interval half width}}$

Therefore, for a two-sided 95% CI we have

$$(\bar{x} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}})$$

which means that the precision is $\boxed{z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}}$

We want $z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} \leq 2$

The length of this interval: $\bar{x} + z_{1-\alpha/2}\sigma/\sqrt{n} - \bar{x} + z_{1-\alpha/2}\sigma/\sqrt{n}$

$= 2z_{1-\alpha/2}\sigma/\sqrt{n}$

$=$

Example:[Width of a CI]

So,

$1-\alpha = 0.95$

$\alpha = 0.05$

$\sigma = 5.1$

$$z_{1-.05/2}\frac{5.1}{\sqrt{n}} \leq 2$$

$$z_{.975}\frac{5.1}{\sqrt{n}} \leq 2$$

$$1.96\frac{5.1}{\sqrt{n}} \leq 2$$

$$\frac{9.996}{\sqrt{n}} \leq 2$$

$$\Rightarrow n \geq 24.98$$

$$\Rightarrow \boxed{n \geq 25}$$

We would need a sample of at least 25 disks to have at least a precision of 2 in.oz

A large-n confidence interval for μ when σ is Unknown

A generally applicable large-n confidence interval for μ

100

Although the equations for a $(1-\alpha)\%$ confidence interval is mathematically correct, it is severely limited in its usefulness because it requres us to know σ (the population variance). It is unusual to have to estimate μ and know σ in real life.

> If $\boxed{n \geq 25}$ and σ is *unknown*, $Z = \frac{\overline{X}-\mu}{s/\sqrt{n}}$, where
>
> $$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2}.$$

is still **approximately standard normally distributed**.

So, you can replace σ in the confidence interval formula with the sample standard deviation, s.

Large Sample Inference

Confidence Interval

CI for μ

CI for μ unknown σ^2

A generally applicable large-n confidence interval for μ

- *Two-sided* $100(1-\alpha)\%$ confidence interval for μ

$$\left(\overline{x} - z_{1-\alpha/2}\frac{s}{\sqrt{n}}, \overline{x} + z_{1-\alpha/2}\frac{s}{\sqrt{n}}\right)$$

- *One-sided* $100(1-\alpha)\%$ confidence interval for μ with a upper confidence bound

$$\left(-\infty\ ,\ \overline{x} + z_{1-\alpha}\frac{s}{\sqrt{n}}\right)$$

- *One-sided* $100(1-\alpha)\%$ confidence interval for μ with a lower confidence bound

$$\left(\overline{x} - z_{1-\alpha}\frac{s}{\sqrt{n}}\ ,\ +\infty\right)$$

Large Sample Inference

Confidence Interval

CI for μ

CI for μ unknown σ^2

Example:

Suppose you are a manufacturer of construction equipment. You make 0.0125 inch wire rope and need to determine how much weight it can hold before breaking so that you can label it clearly. Here are breaking strengths, in kg, for 41 sample wires:

```
 [1] 100.37  96.31  72.57  88.02 105.89
 [6] 107.80  75.84  92.73  67.47  94.87
[11] 122.04 115.12  95.24 119.75 114.83
[16] 101.79  80.90  96.10 118.51 109.66
[21]  88.07  56.29  86.50  57.62  74.70
[26]  92.53  86.25  82.56  97.96  94.92
[31]  62.00  93.00  98.44 119.37 103.70
[36]  72.40  71.29 107.24  64.82  93.51
[41]  86.97
```

$\bar{x}$

> The sample mean breaking strength is 91.85 kg and the sample standard deviation is 17.6 kg.

s

Example: Using the appropriate 95% confidence interval, try to determine whether the breaking strengths meet the requirement of at $\boxed{\text{least 85 kg.}}$

$$(1-\alpha = .95 \quad , \quad \overline{x} = 91.85 \quad , \quad s = 17.6 \quad , \quad n = 41)$$

The CI is then

$$
\begin{aligned}
&\left(\overline{x} - z_{1-\alpha/\cancel{2}} \frac{s}{\sqrt{n}} \ , \ +\infty\right) \\
&= \left(91.85 - z_{.95} \frac{17.6}{\sqrt{41}} \ , \ +\infty\right) \\
&= \left(91.85 - 1.64 \frac{17.6}{\sqrt{41}} \ , \ +\infty\right) \\
&= (87.3422 \ , \ +\infty)
\end{aligned}
$$

> With 95% confidence, we have shown that the true mean breaking strength is above 87.3422 kg.
> Hence, we meet the 85kg requirement with 95% confidence

Small-sample Confidence Interval for a Mean

Small-sample confidence intervals for a mean

- The most important practical limitation on the use of the methods of the previous sections is the requirement that n **must be large** (≥ 25)
- That restriction comes from the fact that without it, there is no way (in general) to calculate $\dfrac{\overline{X}-\mu}{S/n}$ that is approximately $N(0,1)$. (i.e we cannot use CLT when sample size is small)
- So, if one mechanically uses the large- n interval formula $\overline{x} \pm z\frac{s}{\sqrt{n}}$ with a small sample, there is no way of assessing what actual level of confidence should be declared.

Small-sample confidence intervals for a mean

- **If** it is sensible to model the observations as iid normal random variables, then we can arrive at inference methods for small-~~$~~n~~$~~ sample means.

In this case (small sample size), $\overline{x} \pm z\frac{s}{\sqrt{n}}$ is not standard Normal anymore, BUT it is a different normed distribution!

The Student t Distribution

t Student distribution

Definition: The (Student) t distribution with degrees of freedom parameter ν is a continuous probability distribution with probability density

$$f(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}}\left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2} \qquad \text{for all } t \in R.$$

The t distribution

> - is bell-shaped and symmetric about 0
> - has fatter tails than the normal, but approaches the shape of the normal as $\nu \to \infty$.

(heavy tails)

$\Gamma(x) = (x!$

Large Sample Inference

Confidence Interval

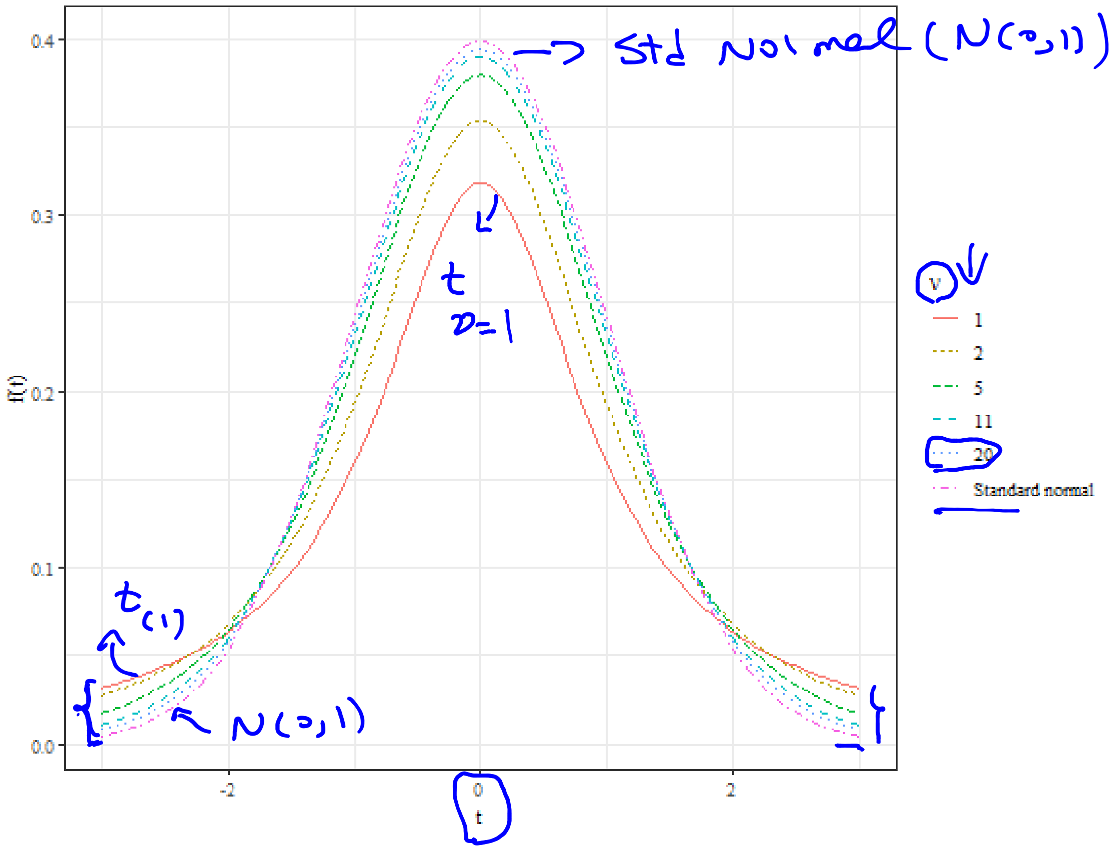
CI for μ

CI for μ unknown σ^2

t Distribution

t Student distribution

We use the t table (Table B.4 in Vardeman and Jobe) to calculate quantiles.



Large Sample Inference

Confidence Interval

CI for μ

CI for μ unknown σ^2

t Distribution

t Student distribution

Example: Say $T \sim t_5$. Find c such that $P(T \leq c) = 0.9$.

ν

Table B.4
t Distribution Quantiles

ν	$Q(.9)$	$Q(.95)$	$Q(.975)$	$Q(.99)$	$Q(.995)$	$Q(.999)$	$Q(.9995)$
1	3.078	6.314	12.706	31.821	63.657	318.317	636.607
2	1.886	2.920	4.303	6.965	9.925	22.327	31.598
3	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869

So, $P(T \leq c) = 0.9$ holds true if $c = 1.476$ (by the table).

$P(T \leq 1.476) = 0.9$

Small-sample Confidence Interval for μ when σ is unknown

Small-sample confidence intervals, σ unknown

If we can assume that $X_1, \ldots, X_n$ are iid with mean μ and variance σ^2, and are also normally distributed, if $n < 25$, we cannot use CLT.

It is not easy to prove but,

$$\frac{\overline{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

We can then use $t_{n-1,1-\alpha/2}$ instead of $z_{1-\alpha/2}$ in the confidence intervals.

Note that the df (degree of freedom) for the t distribution is $n - 1$.

Small-sample confidence intervals, σ unknown

- *Two-sided* $100(1-\alpha)\%$ confidence interval for μ

$$\left(\bar{x} - \boxed{t_{n-1,1-\alpha/2}}\frac{s}{\sqrt{n}}, \bar{x} + t_{n-1,\,1-\alpha/2}\frac{s}{\sqrt{n}}\right)$$

- *One-sided* $100(1-\alpha)\%$ confidence interval for μ with a upper confidence bound

$$\left(-\infty\ ,\ \bar{x} + \boxed{t_{n-1,\,1-\alpha}}\frac{s}{\sqrt{n}}\right)$$

- *One-sided* $100(1-\alpha)\%$ confidence interval for μ with a lower confidence bound

$$\left(\bar{x} - \boxed{t_{n-1,\,1-\alpha}}\frac{s}{\sqrt{n}}\ ,\ +\infty\right)$$

Large Sample Inference

Confidence Interval

CI for μ

CI for μ unknown σ^2

t Distribution

Small n unknown σ^2

Example: [Concrete beams]

10 concrete beams were each measured for flexural strength (MPa). Assuming the flexural strengths are iid ~~normal~~, calculate and interpret a two-sided 99\% CI for the flexural strength of the beams.

$\boxed{\alpha = 0.01}$

[1] 8.2 8.7 7.8 9.7 7.4 7.8 7.7 11.6 11.3 11.8

$$\begin{cases} n = 10 \\ \alpha = 0.01 \end{cases}$$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{10}(8.2 + 8.7 + \dots + 11.8) = 9.2$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$= \frac{1}{10-1}\sum_{i=1}^{10}(x_i - 9.2)^2$$

$$= \frac{1}{9}\left[(8.2-9.2)^2 + (8.7-9.2)^2 + \dots\right.$$

$+ (11.8 - 9.2)^2 \Big] = \cdots$

$$S = \sqrt{S^2} = 1.76$$

$n = 10$

$\alpha = 0.01$

$\bar{x} = 9.2$

$S = 1.76$

$\Rightarrow$ two-sided 99% CI :

$$\left(\bar{x} - t_{(n-1,\, 1-\alpha/2)} \frac{S}{\sqrt{n}},\ \bar{x} + t_{(n-1,\, 1-\alpha/2)} S/\sqrt{n}\right)$$

$$= \left(9.2 - t_{9,\, 0.995} \frac{1.76}{\sqrt{10}},\ 9.2 + t_{9,\, 0.995} \frac{1.76}{\sqrt{10}}\right)$$

$$= \left(9.2 - 3.25 \times \frac{1.76}{\sqrt{10}},\ 9.2 + 3.25 \frac{1.76}{\sqrt{10}}\right)$$

$$= (7.393,\ 11.007)$$

we are 99% confident that the true mean of flexural strength of this kind of beam is between 7.393 and 11.007 MPa.

Large Sample Inference

Confidence Interval

CI for μ

CI for μ unknown σ^2

t Distribution

Small n unknown σ^2

Example: [Concrete beams]

Is the true mean flexural strength below the minimum requirement of 11 MPa? Find out with the appropriate 95% CI.

Find a one-sided CI (upper CI)

$n = 10$

$\alpha = 0.05$

$$\left(-\infty, \bar{x} + t_{(n-1, 1-\alpha)} \frac{s}{\sqrt{n}}\right)$$

$$= \left(-\infty, 9.2 + t_{9, 1-0.05} \frac{1.76}{\sqrt{10}}\right)$$

$$= \left(-\infty, 9.2 + t_{9, 0.95} \cdot \frac{1.76}{\sqrt{10}}\right)$$

$$= \left(-\infty, 9.2 + 1.833 \times \frac{1.76}{\sqrt{10}}\right)$$

$$= (-\infty, 10.22)$$

Interpretation :

we're 95% confident that the true mean flexural strength is below 10.22

$\Rightarrow$ Since this < 11, it meets the requirement.

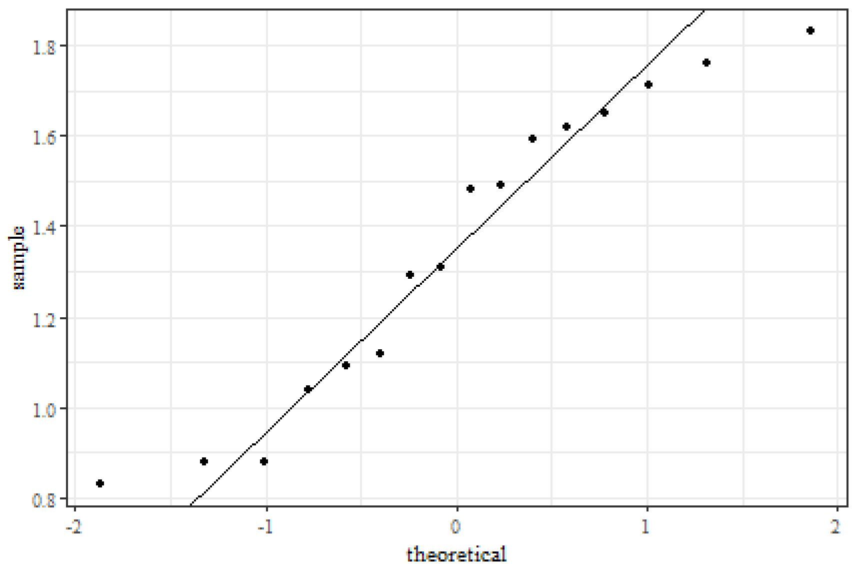
Example: [Paint thickness]

Consider the following sample of observations on coating thickness for low-viscosity paint. (mm)

```
 [1] 0.83 0.88 0.88 1.04 1.09 1.12 1.29
 [8] 1.31 1.48 1.49 1.59 1.62 1.65 1.71
[15] 1.76 1.83
```

$\boxed{n = 16}$

A normal QQ plot shows that they are close enough to normally distributed.



Large Sample Inference

Confidence Interval

CI for μ

CI for μ unknown σ^2

t Distribution

Small n unknown σ^2

Example: [Paint thickness]

Calculate and interpret a two-sided 90% confidence interval for the true mean thickness.

① $n = 16$, $100(1-\alpha)\% = 90\% \Rightarrow \alpha = 0.1$

② $\bar{x} = \frac{1}{16}(0.83 + \cdots + 1.83) = 1.35$ mm

③ $s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$

$= \sqrt{\frac{1}{16-1}\left[(0.83-1.35)^2 + \cdots (1.83-1.35)^2\right]}$

$= 0.34 \boxed{\text{mm}}$

90% CI : $\left(\bar{x} - t_{n-1,1-\alpha/2} \frac{s}{\sqrt{n}}, \bar{x} + t_{n-1,1-\alpha/2} \frac{s}{\sqrt{n}}\right)$

: $\left(1.35 - t_{(15, 1-\frac{0.1}{2})} \frac{0.34}{\sqrt{16}}, 1.35 + t_{(15, 1-\frac{0.1}{2})} \frac{0.34}{\sqrt{16}}\right)$

: $(1.35 - 1.75 \times 0.085, 1.35 + 1.75 \times 0.085)$

$= (1.201, 1.499)$

- we are 90% confident that the true mean tickness of viscosity falls between 1.201 and 1.499.

Let's Wrap Up

Common Assumptions and Common Statements

Suppose that $X_1, X_2, \ldots, X_n$ are random variables whose values will be determined based on the results of random events.

> **Large Sample Size**, **Known Variance**
> Assuming:
>
> - $E(X_i) = \mu$,
> - $n \geq \cancel{30}$, 25
> - $Var(X_i) = \sigma^2$ is known
>
> Then by CLT,
>
> $$* \quad \frac{\bar{X} - \mu}{\sqrt{\sigma^2/n}} \dot{\sim} N(0,1)$$

$100(1-\alpha)\%$ Confidence interval for μ:

$$\bar{x} \pm z_{1-\alpha/2}\sqrt{\frac{\sigma^2}{n}}$$

Common Assumptions and Common Statements

Large Sample Size, **Unknown Variance**

Assuming:

- $E(X_i) = \mu$,
- $n \geq \cancel{20}$, 25
- $Var(X_i)$ is unknown, but sample variance $S^2 = \frac{1}{n-1}\sum (X_i - \bar{X})^2$ can be calculated

Then by CLT and convergence of sample variance

$$\frac{\bar{X} - \mu}{\sqrt{S^2/n}} \sim N(0,1)$$

`

$100 \cdot (1-\alpha)\%$-Confidence interval for μ:

$$\bar{x} \pm z_{1-\alpha/2}\sqrt{\frac{s^2}{n}}$$

Common Assumptions and Common Statements

Small Sample Size, **Unknown Variance**
Assuming:

- $E(X_i) = \mu$,
- $n < \cancel{30}$, 25
- $Var(X_i)$ is unknown, but sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ can be calculated

Then by CLT and convergence of sample variance

$$\frac{\bar{X} - \mu}{\sqrt{S^2/n}} \sim t_{n-1}$$

`

$100 \cdot (1-\alpha)\%$-Confidence interval for μ:

$$\bar{x} \pm t_{(n-1,\, 1-\alpha/2)} \sqrt{\frac{s^2}{n}}$$

Common Assumptions and Common Statements

With the last set of assumptions, we can conclude that $\frac{\bar{X}-\mu}{\sqrt{S^2/n}}$ follows a "t-distribution with $n-1$ degrees of freedom"

The t-distribution looks a lot like a standard normal distribution and we use it the same way:

- It is symmetric
- It is centered at 0
- Important quantiles are collected together in tables for reference

It only has one parameter, the degrees of freedom. In this class, the degrees of freedom are related to the number of parameters being tested

* degrees of freedom = (# of observations) - (# of parameters)

n

1